SYSTEMS FORMED BY TRANSLATES OF ONE ELEMENT IN $L_p(\mathbb{R})$

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ABSTRACT. Let $1 \le p < \infty$, $f \in L_p(\mathbb{R})$ and $\Lambda \subseteq \mathbb{R}$. We consider the closed subspace of $L_p(\mathbb{R})$, $X_p(f,\Lambda)$, generated by the set of translations $f_{(\lambda)}$ of f by $\lambda \in \Lambda$. If $p = 1$ and $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is a bounded minimal system in $L_1(\mathbb{R})$, we prove that $X_1(f,\Lambda)$ embeds almost isometrically into ℓ_1. If $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is an unconditional basic sequence in $L_p(\mathbb{R})$, then $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is equivalent to the unit vector basis of ℓ_p for $1 \le p \le 2$ and $X_p(f,\Lambda)$ embeds into ℓ_p if $2 < p \le 4$. If $p > 4$, there exists $f \in L_p(\mathbb{R})$ and $\Lambda \subseteq \mathbb{Z}$ so that $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is unconditional basic and $L_p(\mathbb{R})$ embeds isomorphically into $X_p(f,\Lambda)$.

1. Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ and $\lambda \in \mathbb{R}$. We denote by $f_{(\lambda)}$ the translation of f λ-units to the right for $\lambda > 0$ (and $|\lambda|$-units to the left for $\lambda < 0$). Precisely,

$$f_{(\lambda)}(x) = f(x - \lambda) \quad \text{for} \quad x \in \mathbb{R} \ .$$

If $f \in L_p(\mathbb{R})$, $1 \le p < \infty$ and $\Lambda \subseteq \mathbb{R}$, we let $X_p(f,\Lambda)$ equal $[\{f_{(\lambda)} : \lambda \in \Lambda\}]$, where $[\cdot]$ denotes the closed linear span in $L_p(\mathbb{R})$. Our main focus shall be on the nature of such subspaces given that $\{f_{(\lambda)} : \lambda \in \Lambda\}$ has some additional structure and Λ is *uniformly discrete*, i.e.,

$$\inf\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda,\ \lambda \ne \lambda'\} > 0 \ .$$

The "additional structure" takes several forms: $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is a bounded minimal system, or is unconditional basic or can be ordered to be a (Schauder) basis or a (Schauder) frame for $X_p(f,\Lambda)$. It is worth mentioning that it is known that if $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is a bounded minimal system, in particular, if it can be ordered to be basic, then Λ must be uniformly discrete. This is easy (Proposition 1.8 below).

The nature of $X_p(f,\Lambda)$ and $\{f_{(\lambda)} : \lambda \in \Lambda\}$ have been studied in a number of papers, mainly using techniques of harmonic analysis. Our techniques will be, largely, from the geometry of Banach spaces. We recall seven theorems, beginning with Wiener's famous Tauberian theorem.

Theorem 1.1. [Wi]. *For $f \in L_2(\mathbb{R})$, $X_2(f,\mathbb{R}) = L_2(\mathbb{R})$ if and only if $\hat{f}(t) \ne 0$ a.e. For $f \in L_1(\mathbb{R})$, $X_1(f,\mathbb{R}) = L_1(\mathbb{R})$ if and only if $\hat{f}(t) \ne 0$ for all $t \in \mathbb{R}$.*

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Theorem 1.2. [AO, Theorem 2.1]. *Let $2 < p < \infty$. There exists $f \in L_p(\mathbb{R})$, all of whose derivatives exist and are in $L_2(\mathbb{R})$ (i.e., $f \in H^{2,\infty}(\mathbb{R})$) so that $X_p(f, \mathbb{Z}) = L_p(\mathbb{R})$. Moreover, f can be chosen to satisfy, in addition, any one of the following conditions.*

(1) $X_p(f, \mathbb{N}_0) = L_p(\mathbb{R})$.
(2) $(f_{(n)})_{n\in\mathbb{Z}}$ *is orthogonal in* $L_2(\mathbb{R})$.
(3) $(f_{(n)})_{n\in\mathbb{Z}}$ *is a bounded minimal system.*

Theorem 1.3. [AO]. *Let $1 \le p \le 2$ and let $F \subseteq L_p(\mathbb{R})$ be a finite set. Then $[\{f_{(n)} : f \in F, n \in \mathbb{Z}\}] \ne L_p(\mathbb{R})$.*

Theorem 1.4. [ER, Corollary 2.11]. *Let $1 \le p < \infty$, $0 \ne f \in L_p(\mathbb{R})$. Then $\{f_{(\lambda)} : \lambda \in \mathbb{R}\}$ is linearly independent.*

Theorem 1.5. [Ol]. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subseteq \mathbb{R}$ so that $\Lambda \cap \mathbb{Z} = \emptyset$ and $\lim_{|n|\to\infty} |\lambda_n - n| = 0$. Then there exists $f \in L_2(\mathbb{R})$ so that $X_2(f, \Lambda) = L_2(\mathbb{R})$.*

Theorem 1.6. [OZ, Theorem 2]. *There is no unconditional basis of translates of f, $\{f_{(\lambda)} : \lambda \in \Lambda\}$, with $X_2(f, \Lambda) = L_2(\mathbb{R})$.*

So the space $X_p(f, \Lambda)$, Λ uniformly discrete, can equal $L_p(\mathbb{R})$, for $p \ge 2$ at least. For $p = 1$ the situation is different as pointed out to us by J. Bruna.

Theorem 1.7. *Let $f \in L_1(\mathbb{R})$ and let $\Lambda \subseteq \mathbb{R}$ be uniformly discrete. Then $X(f, \Lambda) \ne L_1(\mathbb{R})$.*

This seems to be a folk theorem and we were unable to find a reference. It follows from Theorem 1.1 and Lemma 3.3 below (and can also be deduced from [BOU] and the proof of Lemma 3.3).

For $1 < p < \infty$ it remains an open problem whether there exists $\Lambda \subseteq \mathbb{R}$ and $f \in L_p(\mathbb{R})$ so that, in some order, $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is a basis for $L_p(\mathbb{R})$.

In section 2 we prove that if $f \in L_1(\mathbb{R})$ and $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is a bounded minimal system for $X_1(f, \Lambda)$, then $X_1(f, \Lambda)$ embeds almost isometrically into ℓ_1. The same conclusion holds if Λ is uniformly discrete and $\{f_{(\lambda)} : \lambda \in \Lambda\}$ can be ordered to be a (Schauder) frame for $X_1(f, \Lambda)$.

In Corollary 2.10 we show that for $1 \le p \le 2$, if $(f_{(\lambda)})_{\lambda\in\Lambda}$ is an unconditional basic sequence then $(f_{(\lambda)})_{\lambda\in\Lambda}$ is equivalent to the unit vector basis of ℓ_p. For $2 < p \le 4$ we show (Theorem 2.11) that if $(f_{(\lambda)})_{\lambda\in\Lambda}$ is unconditional basic then $X_p(f, \Lambda)$ embeds isomorphically into ℓ_p but (Proposition 2.15) $(f_{(\lambda)})_{\lambda\in\Lambda}$ need not be equivalent to the unit vector basis of ℓ_p. For $4 < p < \infty$ we give an example (Theorem 2.14) of an unconditional basic sequence

$(f_{(\lambda)})_{\lambda\in\Lambda}$ so that $X_p(f,\Lambda)$ contains an isomorph of $L_p[0,1]$ (which, of course, is isometric to $L_p(\mathbb{R})$).

Among further results in section 2, we also consider the translation problem for the translation invariant space $L_p(\mathbb{R})\cap L_2(\mathbb{R})$, $2<p<\infty$, and show (Proposition 2.18) that if $(f_{(\lambda)})_{\lambda\in\Lambda}$ is unconditional basic then it is equivalent to the unit vector basis of ℓ_2.

In the beginning of section 3, we revisit the problem for integer translates of $f\in L_p(\mathbb{R})$. We also prove that if $f\in L_1(\mathbb{R})$ with $\hat{f}(t)\neq 0$ for all t, then $X_1(f,\mathbb{Z})$ embeds into ℓ_1 (Proposition 3.4). We also consider discrete versions of our problem for $\ell_p(\mathbb{Z},X)$ in Propositions 3.5, 3.7 and Corollary 3.12. Fourier analysis plays a role in some of these results.

In section 4, we recall some additional known results from the literature and list some remaining open problems.

We use standard Banach space notation as may be found in [LT] or [JL]. Background material on bases, unconditional bases and such can be found there. For the benefit of those less familiar with these notions we recall some definitions and facts. A *biorthogonal system* is a sequence $(x_i,x_i^*)_{i=1}^\infty\subseteq X\times X^*$ where $x_i^*(x_j)=\delta_{(i,j)}$. A biorthogonal system $(x_i,x_i^*)_{i=1}^\infty\subseteq X\times X^*$ is *fundamental* (or *complete*) if $[(x_i)_{i\in\mathbb{N}}]=X$ and *bounded* if $\sup_i\|x_i\|\,\|x_i^*\|<\infty$.

$(x_i)_{i=1}^\infty\subseteq X$ is a *minimal system* if there exists $(x_i^*)_{i=1}^\infty\subseteq X^*$ so that $(x_i,x_i^*)_{i=1}^\infty$ is a biorthogonal system. This is equivalent to $x_i\notin[x_j : j\neq i]$ for all $i\in\mathbb{N}$. $(x_i)_{i=1}^\infty$ is a *bounded minimal system* if, in addition, $(x_i,x_i^*)_{i=1}^\infty$ is a bounded biorthogonal system. This is equivalent to $\inf_i\operatorname{dist}(x_i,[x_j : j\neq i])>0$. $(x_i)_{i=1}^\infty\subseteq X$ is a (Schauder) *basis* for X if for all $x\in X$ there exists a unique sequence of scalars $(a_i)_{i=1}^\infty$ so that $x=\sum_{i=1}^\infty a_ix_i$. This is equivalent to saying that all $x_i\neq 0$, $[(x_i)]=X$ and for some $K<\infty$, all $m<n$ in $\mathbb{N}$ and all $(a_i)_1^n\subseteq\mathbb{R}$, $\|\sum_{i=1}^m a_ix_i\|\le K\|\sum_{i=1}^n a_ix_i\|$. The smallest such K is the *basis constant* of (x_i). A basis $(x_i)_{i=1}^\infty$ for X is a fundamental bounded minimal system for X. In this case every $x\in X$ can be written uniquely as $x=\sum_{i=1}^\infty x_i^*(x)x_i$. The x_i^*'s are a *basic sequence* in X^*, i.e., form a basis for $[(x_i^*)]\subseteq X^*$ and are a basis for X^* if X is reflexive. $(x_i)_{i=1}^\infty$ is an *unconditional basis* for X if for all $x\in X$ there exists a unique sequence of scalars $(a_i)_{i=1}^\infty$ so that $x=\sum_{i=1}^\infty a_ix_i$ and the *convergence is unconditional*, i.e., $x=\sum_{i=1}^\infty a_{\pi(i)}x_{\pi(i)}$ for all permutations π of $\mathbb{N}$. This is equivalent to all x_i's $\neq 0$, $[(x_i)_{i\in\mathbb{N}}]=X$ and

$$\sup\left\{\Big\|\sum_{i=1}^\infty \varepsilon_ia_ix_i\Big\| : \sum_{i=1}^\infty a_ix_i\in B_X \text{ and } \varepsilon_i=\pm1 \text{ for all } i\right\}<\infty\ .$$

Here B_X denotes the closed unit ball of X. This number is called the *unconditional basis* constant of $(x_i)_{i=1}^\infty$. The biorthogonal functionals then form an unconditional basic sequence in X^*.

A *block basis* $(y_i)_{i=1}^\infty$ of a basic sequence $(x_i)_{i=1}^\infty$ is a non-zero sequence given by

$$y_i = \sum_{j=n_{i-1}+1}^{n_i} a_j x_j \quad \text{for some sequence} \quad n_0 < n_1 < n_2 < \cdots$$

in $\mathbb{N}_0$ and scalars $(a_j)_{j=1}^\infty \subseteq \mathbb{R}$. A block basis is a basic sequence, which is unconditional basic if the x_i's are unconditional basic. A sequence (x_i) is semi-normalized if $0 < \inf \|x_i\| \leq \sup_i \|x_i\| < \infty$.

A *Schauder frame* for a Banach space X is a sequence $(x_i, f_i) \subseteq X \times X^*$ such that for all $x \in X$, $x = \sum_{i=1}^\infty f_i(x) x_i$. Of course every basis for X is a frame for X and just as in the basis case, the uniform boundedness principle yields $\sup\{\|\sum_1^n f_i(x)x_i\| : n \in \mathbb{N},\ x \in S_X\} < \infty$ (called the *frame constant*) where $S_X = \{x \in X : \|x\| = 1\}$ is the unit sphere of X. More on frames can be found in [CHL] and [CDOSZ]. Schauder frames should not be confused with Hilbert frames which are much more restrictive. Note that for frames, (x_i, f_i) is not assumed to be a biorthogonal sequence.

In our situation, where we are concerned with $(f_i)_{i=1}^\infty$ being a sequence of uniformly discrete translations of some $f \in L_p(\mathbb{R})$, we do not know of an example where (f_i) is a frame but is not basic. However, many of our results would hold only given the property of Proposition 2.1 below and so we have stated them in terms of frames.

Some background material on L_p spaces which we shall use can be found in [AOd] and in the basic concepts chapter of [JL]. In particular we shall use that a normalized unconditional basic sequence (f_i) in $L_p(\mathbb{R})$ satisfies for constants A_p and B_p, depending on p and the unconditional basis constant of (f_i),

(1.1) For $1 \leq p \leq 2$, for all $(a_i) \subseteq \mathbb{R}$,

$$A_p^{-1}\Big(\sum_{i=1}^\infty a_i^2\Big)^{1/2} \leq \Big\|\sum_{i=1}^\infty a_i f_i\Big\|_p \leq B_p\Big(\sum_{i=1}^\infty |a_i|^p\Big)^{1/p}.$$

(1.2) For $2 < p < \infty$, $(a_i) \subseteq \mathbb{R}$,

$$A_p^{-1}\Big(\sum_{i=1}^\infty |a_i|^p\Big)^{1/p} \leq \Big\|\sum_{i=1}^\infty a_i f_i\Big\|_p \leq B_p\Big(\sum_{i=1}^\infty |a_i|^2\Big)^{1/2}.$$

If (f_i) is unconditional basic in $L_p[0,1]$, $1 \leq p < \infty$ then for some C_p, depending on p and the unconditional basis constant of (f_i), for all $(a_i) \subseteq \mathbb{R}$,

(1.3) (Square function inequality)

$$\Big\|\sum_{i=1}^\infty a_i f_i\Big\|_p \overset{C_p}{\sim} \left(\int_0^1 \Big(\sum_{i=1}^\infty |a_i|^2 |f_i(t)|^2\Big)^{p/2} dt\right)^{1/p}.$$

Here we use "$A \overset{C}{\sim} B$" to denote $A \le CB$ and $B \le CA$.

The *Haar basis* $(h_n)_{n=1}^\infty$ is a basis for $L_1[0,1]$. This sequence is given by

$$(h_n)_{n=1}^\infty = (\mathcal{X}_{[0,1]},\ \mathcal{X}_{[0,1/2]} - \mathcal{X}_{[1/2,1]},\ \mathcal{X}_{[0,1/4]} - \mathcal{X}_{[1/4,1/2]},\\ \mathcal{X}_{[1/2,3/4]} - \mathcal{X}_{[3/4,1]},\ \mathcal{X}_{[0,1/8]} - \mathcal{X}_{[1/8,1/4]}, \ldots)$$

The same system is an unconditional basis for $L_p[0,1]$, $1 < p < \infty$. Usually below, we will let $(h_n)_{n=1}^\infty$ refer to the normalized Haar basis, i.e., $(h_n/\|h_n\|_p)_{n=1}^\infty$. We can get an unconditional basis for $L_p(\mathbb{R})$ from this by copying $(h_n)_{n=1}^\infty$ onto each interval $[k, k+1]$, $k \in \mathbb{Z}$. In this case we will have functions $(h_{n,k})_{n\in\mathbb{N},\ k\in\mathbb{Z}}$ and we will presume they are *linearly ordered* so as to be *compatible* with the Haar basis ordering on each $[k, k+1]$, i.e., if the functions are ordered as $(x_i)_{i=1}^\infty$ and if $x_i = h_{n,k}$, $x_j = h_{m,k}$ with $i < j$, then $n < m$. This ordering yields that if $(g_i)_{i=1}^\infty$ is a block basis of the Haar basis, then $(g_i|_{[n,m]})_{i=1}^\infty$ is also a block basis of the Haar basis (well, some g_i's could be 0 here) for all integers $n < m$.

The *Rademacher sequence* $(r_n)_{n=1}^\infty$ is given by $(r_n)_{n=1}^\infty = (h_1, h_2, h_3+h_4, h_5+\cdots+h_8, \ldots)$, where the h_n's refer to the non-normalized Haar functions. It is *equivalent to the unit vector basis of* ℓ_2 in all $L_p[0,1]$ spaces, $1 \le p < \infty$, i.e.,

$$\Big\| \sum_{i=1}^\infty a_i r_i \Big\|_p \overset{K_p}{\sim} \Big(\sum_{i=1}^\infty |a_i|^2 \Big)^{1/2}.$$

One reason for taking Λ to be uniformly discrete in our considerations is, as mentioned above, given by the easy

Proposition 1.8. [OZ, Theorem 1]. *Let $1 \le p < \infty$ and let $(f_i, g_i)_{i=1}^\infty$ be a bounded biorthogonal system in $L_p(\mathbb{R})$ so that for some $f \in L_p(\mathbb{R})$ and $(\lambda_i)_1^\infty \subseteq \mathbb{R}$, $f_i = f_{(\lambda_i)}$ for all i. Then $\Lambda = (\lambda_i)_1^\infty$ is uniformly discrete.*

Proof. If not, there exist subsequences (i_m) and (j_m) of $\mathbb{N}$ so that $\lim_{m\to\infty} |\lambda_{i_m} - \lambda_{j_m}| = 0$ and $\lambda_{i_m} \ne \lambda_{j_m}$ for all m. Then

$$\|g_{i_m}\| \ge \frac{\langle g_{i_m}, f_{i_m} - f_{j_m}\rangle}{\|f_{i_m} - f_{j_m}\|_p} = \frac{1}{\|f_{i_m} - f_{j_m}\|_p}$$

and the latter is unbounded in m, a contradiction. □

2. Main Results

We begin with the elementary but very useful

Proposition 2.1. *Let $\Lambda \subseteq \mathbb{R}$ be uniformly discrete, $1 \le p < \infty$, and $f \in L_p(\mathbb{R})$. Then for all intervals $I = [a,b]$, $\sum_{\lambda\in\Lambda} \|f_{(\lambda)}|_I\|_p^p < \infty$.*

Proof. Choose $\varepsilon_0 > 0$ so that $|\lambda - \lambda'| > \varepsilon_0$ for all $\lambda, \lambda' \in \Lambda$ with $\lambda \neq \lambda'$. For $\ell \in \mathbb{Z}$, set $I_\ell = [a + (\ell-1)\varepsilon_0, a + \ell\varepsilon_0]$. Then for $\ell \in \mathbb{Z}$,

$$\sum_{\lambda\in\Lambda} \|f_{(\lambda)}|_{I_\ell}\|_p^p = \sum_{\lambda\in\Lambda} \int_{I_\ell} |f(t-\lambda)|^p\,dt = \sum_{\lambda\in\Lambda} \int_{a+(\ell-1)\varepsilon_0-\lambda}^{a+\ell\varepsilon_0-\lambda} |f(t)|^p\,dt \le \|f\|_p^p ,$$

since the intervals of integration are disjoint for $\lambda \in \Lambda$. Thus

$$\sum_{\lambda\in\Lambda} \|f_{(\lambda)}|_I\|_p^p \le \sum_{\ell=1}^{\left\lceil \frac{b-a}{\varepsilon_0} \right\rceil} \sum_{\lambda\in\Lambda} \|f_{(\lambda)}|_{I_\ell}\|_p^p \le \left\lceil \frac{b-a}{\varepsilon_0} \right\rceil \|f\|_p^p . \qquad \square$$

We note a simple consequence of Proposition 2.1. We remark that in [AO, Theorem 4.1], it is proved that if $1 < p < \infty$ and $f \in L_p(\mathbb{R}) \cap L_1(\mathbb{R})$ then $X_p(f, \mathbb{Z}) \neq L_p(\mathbb{R})$.

Proposition 2.2. *Let $1 < p < \infty$, $f \in L_p(\mathbb{R}) \cap L_1(\mathbb{R})$, and let $(f_i)_{i=1}^\infty$ be a sequence of uniformly discrete translates of f. Then $(f_i)_{i=1}^\infty$ is not a fundamental bounded minimal system for $L_p(\mathbb{R})$. Furthermore, there is no sequence $(g_i)_{i=1}^\infty \subseteq L_q(\mathbb{R})$ $(1/p + 1/q = 1)$ so that $(f_i, g_i)_{i=1}^\infty$ is a frame for $L_p(\mathbb{R})$.*

Proof. Assume (f_i, g_i) were in fact such a frame. $\|f_i\|_p = \|f\|_p$ for all i and thus $(g_i)_{i=1}^\infty$ is ω^*-null and hence bounded in $L_q(\mathbb{R})$. Let $K = \sup_i \|g_i\|_q$. Choose $n_0 \in \mathbb{N}$ with

$$\sum_{j=n_0+1}^{\infty} \|f_i|_{[0,1]}\|_1 < \frac{1}{4K} .$$

Choose $h : \mathbb{R} \to \mathbb{R}$ so that $|h| = \chi_{[0,1]}$ and $|\langle h, g_i \rangle| < \frac{1}{4n_0\|f\|_1}$ for $i \le n_0$ (h could be a Rademacher function). Thus $\|h\|_p = \|h\|_1 = 1$. Also $h = \sum_{i=1}^\infty \langle h, g_i \rangle f_i$, the series converging in $L_p(\mathbb{R})$, and so

$$h|_{[0,1]} = \sum_{i=1}^{\infty} \langle h, g_i \rangle f_i|_{[0,1]} ,$$

the series converging in $L_1[0,1]$. Then

$$\begin{aligned} 1 = \|h\|_1 &\le \sum_{i=1}^{\infty} |\langle h, g_i \rangle| \|f_i|_{[0,1]}\|_1 \\ &\le \sum_{i=1}^{n_0} |\langle h, g_i \rangle| \|f\|_1 + \sum_{i=n_0+1}^{\infty} \|g_i\|_q \|f_i|_{[0,1]}\|_1 \\ &< \frac{n_0}{4n_0\|f\|_1} \|f\|_1 + \sup_i \|g_i\|_q \frac{1}{4K} = \frac{1}{2} , \end{aligned}$$

a contradiction.

The argument is similar if we assume that $(f_i, g_i)_{i=1}^\infty$ is a fundamental bounded biorthogonal system for $L_p(\mathbb{R})$. Then, for the same h, n_0 and for $\varepsilon > 0$ arbitrary, we can choose

$f = \sum_{i=1}^n a_i f_i$ with $\|h - f\|_p < \varepsilon$. Thus $\|f|_{[0,1]} - h\|_1 < \varepsilon$ and

$$1 - \varepsilon \le \|f|_{[0,1]}\|_1 \le \sum_{i=1}^{n_0} |a_i|\,\|f_i|_{[0,1]}\|_1 + \sum_{i=n_0+1}^{n} |a_i|\,\|f_i|_{[0,1]}\|_1\ . \tag{2.1}$$

For $i \le n_0$,

$$|a_i| = |g_i(f)| \le |g_i(f-h)| + |g_i(h)| < K\varepsilon + \frac{1}{4n_0\|f\|_1}\ .$$

For $i > n_0$, $|a_i| \le K(1+\varepsilon)$. Hence by (2.1)

$$\begin{aligned} 1 - \varepsilon &\le n_0\left(K\varepsilon + \frac{1}{4n_0\|f\|_1}\right)\|f\|_1 + \sum_{i=n_0+1}^{n} K(1+\varepsilon)\|f_i|_{[0,1]}\|_1 \\ &< n_0K\varepsilon\|f\|_1 + \frac{1}{4} + \frac{1}{4}(1+\varepsilon) < \frac{3}{4} < 1 - \varepsilon\ , \end{aligned}$$

a contradiction, if $\varepsilon < 1/4$. $\square$

For $p = 1$ we have a stronger result as a consequence of our next theorem (Corollary 2.4).

Definition. Let $1 \le p < \infty$, $1/p + 1/q = 1$.

a) Let $(f_i, g_i) \subseteq L_p(\mathbb{R}) \times L_q(\mathbb{R})$ be a frame for a subspace X of $L_p(\mathbb{R})$. We say (f_i, g_i) satisfies $(*)$ if

$(*)$ for all $\varepsilon > 0$ and all bounded intervals $I \subseteq \mathbb{R}$, there exists $n \in \mathbb{N}$ so that for all $m \ge n$ and $f \in X$,

$$\Big\| \sum_{i=m+1}^{\infty} \langle f, g_i\rangle f_i|_I \Big\|_p \le \varepsilon\|f\|_p\ .$$

b) A semi-normalized bounded minimal system $(f_i)_{i=1}^\infty$ in $L_p(\mathbb{R})$ satisfies $(**)$ if

$(**)$ for all $\varepsilon > 0$ and bounded intervals $I \subseteq \mathbb{R}$ there exists $n \in \mathbb{N}$ so that for all $n < m \le m_1 \le m_2$ and $f = \sum_{i=1}^{m_2} a_i f_i$ with $\|f\|_p = 1$, $\|\sum_{i=m}^{m_1} a_i f_i|_I\| \le \varepsilon$.

Theorem 2.3. *Let $(f_i, g_i)_{i=1}^\infty$ be a frame or a semi-normalized bounded fundamental minimal system for a subspace X of $L_p(\mathbb{R})$, $1 \le p < \infty$, satisfying $(*)$ or $(**)$, respectively. Then X embeds almost isometrically into ℓ_p.*

X embeds *almost isometrically into ℓ_p* means that for all $\varepsilon > 0$ there exits $T : X \to \ell_p$ with $(1+\varepsilon)^{-1} \le \|Tf\| \le 1 + \varepsilon$ for all $f \in S_X$. The proof of Theorem 2.3 will yield, for all $\varepsilon > 0$, a partition $\Pi = (D_s)_{s=1}^\infty$ of $\mathbb{R}$ into intervals so that for all $f \in S_X$,

$$\|f - \mathbb{E}_\Pi f\|_p < \varepsilon\ .$$

$\mathbb{E}_\Pi$ is the conditional expectation projection

$$f \longmapsto \sum_{s=1}^{\infty}\left(\int_{D_s} f\right)\frac{\chi_{D_s}}{m(D_s)}\ .$$

Of course, in L_p, $(\frac{\chi_{D_s}}{m(D_s)})$ is 1-equivalent to the unit vector basis of ℓ_p.

From Proposition 2.1 and Theorem 2.3 we obtain

Corollary 2.4. *If (f_i, g_i) is a frame or a fundamental bounded minimal system for a subspace X of $L_1(\mathbb{R})$ where the f_i's are uniformly discrete translates of some $f \in L_1(\mathbb{R})$, then X embeds almost isometrically into ℓ_1.*

Proof of Theorem 2.3. We first consider the frame case and let C be the frame constant. Thus for all $f \in X$ and $n \in \mathbb{N}$,

$$\Big\| \sum_{i=1}^{n} \langle f, g_i \rangle f_i \Big\|_p \le C \|f\|_p \ .$$

Let $\varepsilon > 0$. We inductively choose increasing sequences (m_k) and (n_k) in $\mathbb{N}$ to obtain, where $I_k = [-m_k, m_k]$,

$$\text{for } \ f \in X, \ \text{ and } \ n \ge n_k \ , \quad \Big\| \sum_{i=n+1}^{\infty} \langle g_i, f \rangle f_i|_{I_{k-1}} \Big\|_p \le \varepsilon 2^{-k} \|f\|_p \ , \tag{2.2}$$

$$\text{for } \ f \in \operatorname{span}\{f_i : i \le n_k\} \ , \quad \|f|_{\mathbb{R}\setminus I_k}\|_p \le \varepsilon 2^{-k} \|f\|_p \ . \tag{2.3}$$

We do this by setting $I_0 = \emptyset$, letting n_1 be arbitrary and choose m_1 to satisfy (2.3) for $k = 1$. Then choose n_2 to satisfy (2.2) using $(*)$ and continue in this manner. We let $A_k = I_k \setminus I_{k-1}$, for $k \in \mathbb{N}$.

Choose a partition π_k of A_k into intervals, $k \ge 1$, so that for all $f \in \operatorname{span}\{f_i : i \le n_{k+1}\}$,

$$\Big\| f|_{A_k} - \sum_{D \in \pi_k} \frac{\chi_D}{m(D)} \int_D f(x)\, dx \Big\|_p \le \varepsilon 2^{-k} \|f\|_p \ . \tag{2.4}$$

Let $f \in X$ with $\|f\|_p = 1$. Then, with $n_0 = 0$,

$$1 = \Big\| \sum_{i=1}^{\infty} \langle g_i, f \rangle f_i \Big\|_p = \Big\| \sum_{s=1}^{\infty} \Big(\sum_{i=n_{s-1}+1}^{n_s} \langle g_i, f \rangle f_i|_{I_s} \Big) + \sum_{s=1}^{\infty} \Big(\sum_{i=n_{s-1}+1}^{n_s} \langle g_i, f \rangle f_i|_{\mathbb{R}\setminus I_s} \Big) \Big\|_p$$

$$\le \Big\| \sum_{s=1}^{\infty} \sum_{i=n_{s-1}+1}^{n_s} \langle g_i, f \rangle f_i|_{I_s} \Big\|_p + 2C\varepsilon \ , \quad \text{by (2.3)}$$

$$\le \Big\| \sum_{s=1}^{\infty} \sum_{i=n_{s-1}+1}^{n_s} \langle g_i, f \rangle f_i|_{I_s \setminus I_{s-2}} \Big\|_p + 2C\varepsilon + 2\varepsilon \ , \quad \text{by (2.2)}$$

where we let $I_{-1} = I_0 = \emptyset$

$$= \Big\| \sum_{s=1}^{\infty} \sum_{i=n_{s-1}+1}^{n_s} \langle g_i, f \rangle f_i|_{A_s \cup A_{s-1}} \Big\|_p + 2C\varepsilon + 2\varepsilon$$

where we let $A_0 = \emptyset$

$$= \Big\| \sum_{s=1}^\infty \chi_{A_s} \sum_{i=n_{s-1}+1}^{n_{s+1}} \langle g_i, f\rangle f_i \Big\|_p + 2C\varepsilon + 2\varepsilon$$

$$\leq \Big(\sum_{s=1}^\infty \sum_{D\in\pi_s} \Big| \int_D \sum_{i=n_{s-1}+1}^{n_{s+1}} \langle g_i, f\rangle f_i(x)\,dx \Big|^p \Big)^{1/p} + 4C\varepsilon + 2\varepsilon \ \text{ by (2.4)} . \tag{2.5}$$

Now by (2.2) for $s \in \mathbb{N}$,

$$\Big(\sum_{D\in\pi_s} \Big| \int_D \sum_{i=n_{s+1}+1}^{\infty} \langle g_i, f\rangle f_i(x)\,dx \Big|^p \Big)^{1/p} \leq \Big\| \sum_{i=n_{s+1}+1}^{\infty} \langle g_i, f\rangle f_i|_{I_s} \Big\|_p \leq \varepsilon 2^{-(s+1)} . \tag{2.6}$$

If $s > 1$ then by (2.3),

$$\Big(\sum_{D\in\pi_s} \Big| \int_D \sum_{i=1}^{n_{s-1}} \langle g_i, f\rangle f_i(x)\,dx \Big|^p \Big)^{1/p} \leq 2^{-s+1} C\varepsilon . \tag{2.7}$$

From (2.5), (2.6) and (2.7) we obtain that

$$1 = \|f\|_p \leq \Big(\sum_{D\in \bigcup_{s=1}^\infty \pi_s} \Big| \int_D f(x)\,dx \Big|^p \Big)^{1/p} + \sum_{s=1}^\infty \varepsilon 2^{-(s+1)} + \sum_{s=2}^\infty 2^{-s+1} C\varepsilon + 6C\varepsilon$$

$$\leq \Big(\sum_{D\in \bigcup_{s=1}^\infty \pi_s} \Big| \int_D f(x)\,dx \Big|^p \Big)^{1/p} + 8C\varepsilon = 1 + 8C\varepsilon .$$

Thus $T : X \to \ell_p(\bigcup_{s=1}^\infty \pi_s)$ given by $f \mapsto (\int_D f(x)\,dx)_{D\in\bigcup_{s=1}^\infty \pi_s}$ is the desired embedding.

The proof in the case that $(f_i)_{i=1}^\infty$ is a bounded fundamental minimal system for $X \subseteq L_p(\mathbb{R})$ is nearly identical. We let $K = \sup_i \|g_i\|_q$ and in the construction replace (2.2)–(2.4) by

$$\text{For all } n_k < n \leq m \leq \bar m \text{ and } f = \sum_1^{\bar m} a_i f_i \in S_X , \tag{2.2$'$}$$

$$\Big\| \sum_{i=n}^m a_i f_i|_{I_{k-1}} \Big\|_p \leq \varepsilon 2^{-k} \ \text{ (using (**)).}$$

$$\text{For all } \ f = \sum_{i=1}^{n_k} a_i f_i \ \text{ with } |a_i| \leq K \text{ for } i \leq n_k, \ \ \|f|_{\mathbb{R}\setminus I_k}\|_p \leq \varepsilon 2^{-k} . \tag{2.3$'$}$$

$$\text{For all } \ f = \sum_{i=1}^{n_{k+1}} a_i f_i \ \text{ with } |a_i| \leq K \text{ for } i \leq n_{k+1}, \tag{2.4$'$}$$

$$\Big\| f|_{A_k} - \sum_{D\in\Pi_k} \frac{\chi_D}{m(D)} \int_D f(x)\,dx \Big\|_p \leq \varepsilon 2^{-k} .$$

The proof then proceeds as in the frame case for $f \in \operatorname{span}(f_i)$, $f = \sum_{i=1}^\ell a_i f_i$, $\|f\|_p = 1$. $\square$

Remark 2.5. Let $X \subseteq L_p(\mathbb{R})$ be as in Theorem 2.3 with $1 < p < \infty$. Then there is a shorter proof that yields $X \hookrightarrow \ell_p$. In fact in the bounded minimal system case, one can replace $(**)$ by the weaker

$(***)$ For all $\varepsilon > 0$ and bounded intervals $I \subseteq \mathbb{R}$ there exists $n \in \mathbb{N}$ so that if $f \in \text{span}(f_i)_{i \geq n}$ with $\|f\|_p = 1$, then $\|f|_I\|_p < \varepsilon$.

Indeed by [KP], [J] and [JO], it suffices to prove that if (x_n) is a normalized weakly null sequence in X then some subsequence is 2-equivalent to the unit vector basis of ℓ_p. Then, from $(*)$ or $(***)$, it is easy to find (x_{n_i}) and intervals $I_1 \subseteq I_2 \subseteq \cdots$ so that $\|x_{n_i}|_{I_i \setminus I_{i-1}}\|_p > 1 - \frac{\varepsilon}{2^i}$ for all i and deduce the result.

We will say that a *frame* $(f_i, g_i)_{i=1}^\infty$ *for* X *satisfies a lower* ℓ_q*-estimate* if for some $K < \infty$ and all $x \in X$,

$$\Big(\sum_{i=1}^\infty |g_i(x)|^q\Big)^{1/q} \leq K\Big\|\sum_{i=1}^\infty g_i(x) f_i\Big\| = K\|x\| \ .$$

A Hilbert frame, by definition, satisfies lower (and upper) ℓ_2-estimates.

If $(x_i)_{i=1}^\infty$ is a fundamental bounded minimal system for X we say that $(x_i)_{i=1}^\infty$ satisfies a lower ℓ_q-estimate if for some K and all scalars $(a_i)_1^n$

$$\Big(\sum_{i=1}^n |a_i|^q\Big)^{1/q} \leq K\Big\|\sum_{i=1}^n a_i x_i\Big\| \ .$$

Proposition 2.6. *Let $1 < p < \infty$, $1/p + 1/q = 1$.*

a) *Assume $1 < p \leq 2$ and let $(f_i)_{i=1}^\infty$ be a sequence of uniformly discrete translations of $f \in L_p(\mathbb{R})$. Let either $(f_i, g_i)_{i=1}^\infty \subseteq L_p(\mathbb{R}) \times L_q(\mathbb{R})$ be a frame or $(f_i)_{i=1}^\infty$ be a fundamental bounded minimal system for $X \subseteq L_p(\mathbb{R})$. If $(f_i)_{i=1}^\infty$ admits a lower ℓ_q-estimate, then X embeds almost isometrically into ℓ_p.*

b) *Let $(f_i, g_i)_{i=1}^\infty$ be a frame for $L_p(\mathbb{R})$, where $(f_i)_{i=1}^\infty$ is a sequence of uniformly discrete translations of $f \in L_p(\mathbb{R})$. Then for all bounded measurable sets B of positive measure, $\sum_{i=1}^\infty \|g_i|_B\|_1^q = \infty$.*

Remark 2.7. The hypothesis in a) would be vacuous for $p > 2$ since some subsequence of (f_i) is equivalent to the unit vector basis of ℓ_p.

Proof. First let $(f_i, g_i)_{i=1}^\infty$ be a frame for $X \subseteq L_p(\mathbb{R})$ as in a). Assume for all $f \in X$,

$$\Big(\sum_{i=1}^\infty |g_i(f)|^q\Big)^{1/q} \leq K\|f\|_p \ .$$

For any bounded interval $I \subseteq \mathbb{R}$, $f \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned}\Big\| \sum_{i=n}^{\infty} g_i(f) f_i\big|_I \Big\|_p &\le \sum_{i=n}^{\infty} |g_i(f)|\ \|f_i\big|_I\|_p \\ &\le \Big(\sum_{i=n}^{\infty} |g_i(f)|^q \Big)^{1/q} \Big(\sum_{i=n}^{\infty} \|f_i\big|_I\|_p^p \Big)^{1/p} \\ &\le K\|f\|_p \Big(\sum_{i=n}^{\infty} \|f_i\big|_I\|_p^p \Big)^{1/p} .\end{aligned}$$

From Proposition 2.1 we obtain that $(*)$ holds and so Theorem 2.3 applies.

Similarly, if (f_i) is a fundamental bounded minimal system for X and $f = \sum_1^{\bar{m}} a_i f_i$ with $\|f\|_p = 1$, we have

$$\Big\| \sum_{n}^{m} a_i f_i|_I \Big\|_p \le K \Big(\sum_{n}^{m} \|f_i|_I\|_p \Big)^{1/p}$$

and so, again, we have $(**)$ and apply Theorem 2.3.

b) Assume that for some B of positive measure $\sum_{i=1}^{\infty} \|g_i|_B\|_1^q < \infty$. Let $h \in L_\infty(B)$, $|h| = 1$. So $h = \sum_{i=1}^{\infty} \langle h, g_i \rangle f_i|_B$, the series converging in $L_1(B)$. Thus

$$\begin{aligned}m(B) = \|h\|_1 &\le \sum_{i=1}^{n} |\langle h, g_i\rangle| \|f_i\|_1 + \sum_{i=n+1}^{\infty} |\langle h, g_i|_B \rangle|\ \|f_i|_B\|_1 \\ &\le \sum_{i=1}^{n} |\langle h, g_i\rangle|\ \|f_i\|_1 + \Big(\sum_{i=n+1}^{\infty} \|g_i|_B\|_1^q \Big)^{1/q} \Big(\sum_{i=n+1}^{\infty} \|f_i|_B\|_1^p \Big)^{1/p} .\end{aligned}$$

Then, as in the proof of Proposition 2.2, we can choose n so that the second term does not exceed $m(B)/4$, and given this n, choose h to make the first term also less than $m(B)/4$. Thus $m(B) < \frac{1}{2} m(B)$, a contradiction. $\square$

Part a) of Proposition 2.6 yields a quantitative improvement of Theorem 1.6. If $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is unconditional basic in $L_2(\mathbb{R})$, then given $\varepsilon > 0$ there is a partition Π of $\mathbb{R}$ so that for all $g \in X_2(f, \Lambda)$,

$$\|g - \mathbb{E}_\Pi g\|_2 \le \varepsilon \|g\|_2 .$$

Remark 2.8. a) Let $f = \chi_{[0,1]} - \chi_{[1,2]} \in L_p(\mathbb{R})$. The sequence $(f_{(n)})_{n\in\mathbb{Z}}$ is basic in $L_1(\mathbb{R})$ when ordered as $(f_{(0)}, f_{(1)}, f_{(-1)}, f_{(2)}, f_{(-2)}, \ldots)$. It is not unconditional since

$$\Big\| \sum_{i=-2n}^{2n} f_{(i)} \Big\|_p = 2^{1/p} \quad \text{but} \quad \Big\| \sum_{i=-n}^{n} f_{(2i)} \Big\|_p = (2n+2)^{1/p} .$$

For $1 < p < \infty$, $(f_{(n)})_{n\in\mathbb{Z}}$ is not a frame nor a minimal system in $L_p(\mathbb{R})$. The latter follows easily from the fact that

$$\lim_{n\to\infty} \Big\| f_{(0)} + \sum_{k=1}^{n} \frac{n-k}{n}(f_{(k)} + f_{(-k)})\Big\|_p = 0 \ .$$

b) (due to S.J. Dilworth) Let $1 \le p < \infty$ and let

$$f = \chi_{[-3/2,\,-1/2]} + 2\chi_{[-1/2,\,1/2]} + \chi_{[1/2,\,3/2]} \ .$$

For $n \in \mathbb{N}$ set

$$g_n = f + \sum_{k=1}^{n} (-1)^n \ \frac{n-k+1}{n} \ (f_{(k)} + f_{(-k)}) \ .$$

Then for $x \ge 0$

$$g_n(x) = \begin{cases} -f_{(-1)}(x) + f_{(0)}(x) - f_{(1)}(x) = -1+2-1 = 0 & \text{if } x \in [0, \frac{1}{2}] \\ f_{(0)}(x - f_{(1)}(x) + \frac{n-1}{n} f_{(2)}(x) = 1-2+\frac{n-1}{n} = -\frac{1}{n} & \text{if } x \in [\frac{1}{2}, \frac{3}{2}] \\ (-1)^k \dfrac{n-k+1}{n} f_{(k)}(x) + (-1)^{k+1} \dfrac{n-k}{n} f_{(k+1)}(x) & \\ \qquad\qquad + (-1)^{k+2} \frac{n-k-1}{n} f_{(k+2)}(x) = 0 & \text{if } x \in [k+\frac{1}{2}, k+\frac{3}{2}] \\ & \text{for some } 1 \le k \le n-2 \\ (-1)^{n-1} \dfrac{2}{n} f_{(n-1)}(x) + (-1)^n \dfrac{1}{n} f_{(n)}(x) = 0 & \text{if } x \in [n-\frac{1}{2}, n+\frac{1}{2}] \\ (-1)^n \dfrac{1}{n} f_{(n)}(x) = (-1)^n \dfrac{1}{n} & \text{if } x \in [n+\frac{1}{2}, n+\frac{3}{2}] \ . \end{cases}$$

Thus $\|g_n\|_p = 4^{1/p}/n$, hence $f_{(0)} \in [\{f_{(k)} : k \in \mathbb{Z} \setminus \{0\}\}]$ and so $(f_{(k)})_{k\in\mathbb{Z}}$ is not a minimal system in $L_p(\mathbb{R})$. Furthermore,

$$\hat{\chi}_{[-1/2,\,1/2]}(x) = \frac{1}{\sqrt{2\pi}} \int_{-1/2}^{1/2} e^{-ixt}\, dt = \frac{1}{\sqrt{2\pi}} \ \frac{\sin x}{x} \ .$$

It follows that

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \ \frac{\sin x}{x} [2 + \cos x]$$

so $\hat{f}(x) \ne 0$ a.e..

c) (due to D. Freeman) It is well known that if $(f_i)_{i=1}^{\infty}$ is a normalized sequence in $L_1(\mathbb{R})$ with $\|f_i|_{I_i}\| \ge \lambda > \frac{1}{2}$ for all i and some sequence of pairwise disjoint measurable sets $I_i \subseteq \mathbb{R}$, then $(f_i)_{i=1}^{\infty}$ is equivalent to the unit vector basis of ℓ_1. Indeed

$$\begin{aligned} \|\sum a_i f_i\|_1 &\ge \|\sum a_i f_i|_{I_i}\|_1 - \sum |a_i| \ \|f_i|_{\mathbb{R}_i \setminus I_i}\|_1 \\ &\ge \lambda \sum |a_i| - (1-\lambda) \sum |a_i| = (2\lambda - 1) \sum |a_i| \ . \end{aligned}$$

Thus if $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is a sequence of uniformly discrete translations of $0 \neq f \in L_1(\mathbb{R})$, then it can be split into a finite number of subsequences, each equivalent to the unit vector basis of ℓ_1.

d) By Theorem 1.4, if $1 \leq p < \infty$ and $f \in L_p(\mathbb{R})$, $f \neq 0$ then $\{f_{(\lambda)} : \lambda \in \mathbb{R}\}$ is linearly independent (see also Theorem 4.2 below). However one can find $f \in L_1(\mathbb{R})$ so that $\{f_{(n)} : n \in \mathbb{Z}\}$ is not ω-linearly independent in its natural order [R].

We next turn to the case where (f_i) is unconditional basic in L_p. We first recall

Proposition 2.9. [JO, Lemma 2]. *Let $1 \leq p \leq 2$. Let $(f_i) \subseteq L_p(\mathbb{R})$ be seminormalized and unconditional basic. Assume that for some $\delta > 0$ there exists a sequence of disjoint measurable sets $(B_i)_{i=1}^\infty$ with $\|f_i|_{B_i}\|_p \geq \delta$, for all i. Then $(f_i)_{i=1}^\infty$ is equivalent to the unit vector basis of ℓ_p.*

Corollary 2.10. *Let $(f_i)_{i=1}^\infty$ be an unconditional basic sequence in $L_p(\mathbb{R})$, $1 \leq p \leq 2$. Assume the f_i's are all translates of some fixed $f \in L_p(\mathbb{R})$. Then $(f_i)_{i=1}^\infty$ is equivalent to the unit vector basis of ℓ_p.*

Proof. Let $f_i = f_{(\lambda_i)}$ for $i \in \mathbb{N}$. Let $\rho \equiv \frac{1}{2} \inf\{|\lambda_i - \lambda_j| : i \neq j\} > 0$. Let I be an interval of length ρ with $\|f|_I\|_p = \delta > 0$. If $B_i = I + \lambda_i$ for $i \in \mathbb{N}$ then the B_i's are pairwise disjoint and $\|f_i|_{B_i}\| = \|f|_I\| = \delta$ for all i. Proposition 2.9 yields the result. $\square$

As we shall see the situation is more complicated for $p > 2$, and especially so for $p > 4$.

Theorem 2.11. *Let $2 < p \leq 4$ and let $(f_i) \subseteq L_p(\mathbb{R})$ be an unconditional basis for $X \subseteq L_p(\mathbb{R})$. Assume the f_i's are all translates of some fixed $f \in L_p(\mathbb{R})$. Then X embeds isomorphically into ℓ_p.*

Lemma 2.12. *Let $p \neq 2$ and let X be a subspace of $L_p(\mathbb{R})$ not containing an isomorph of ℓ_p. Then there exists $c > 0$ so that $|||f||| = \|f|_{[-c,c]}\|_p$ is an equivalent norm on X.*

Proof. If the lemma is false then we can find $(f_k)_{k=1}^\infty \subseteq S_X$ and $(m_k)_{k=1}^\infty \subseteq \mathbb{N}$ so that $\|f_k|_{[-m_k,m_k]}\| \geq 1 - 2^{-2k-1}$ and $\|f_{k+1}|_{[-m_k,m_k]}\| \leq 2^{-2k-1}$ for all $k \in \mathbb{N}$. It follows easily that $(f_k)_{k=1}^\infty$ is equivalent to $(f_k|_{[-m_k,m_k]\setminus[-m_{k-1},m_{k-1}]})_{k=1}^\infty$ which, being seminormalized and disjointly supported, is equivalent to the unit vector basis of ℓ_p. $\square$

We shall also use

Proposition 2.13. [JO]. *Let X be a subspace of $L_p(\mathbb{R})$, $2 < p < \infty$, which does not contain an isomorph of ℓ_2. Then X embeds isomorphically into ℓ_p.*

In fact by [KW], X must then embed almost isometrically into ℓ_p.

We set some notation and recall some things before proving the theorem. We let (h_i) denote the normalized Haar basis for $L_p[0,1]$ regarded, canonically, as a subspace of $L_p(\mathbb{R})$. As mentioned in the introduction, for $i \in \mathbb{N}$ and $n \in \mathbb{Z}$ we let $h_{(i,n)}(\cdot) = h_i((\cdot) - n)$. Thus, $(h_{(i,n)})_{i\in\mathbb{N},\, n\in\mathbb{Z}}$ is an unconditional basis for $L_p(\mathbb{R})$.

G. Schechtman [S] made the very useful observation that if $(f_i)_{i=1}^\infty$ and $(g_i)_{i=1}^\infty$ are seminormalized unconditional basic sequences in $L_p(\mathbb{R})$, $1 < p < \infty$, with

$$\sum_{i=1}^\infty \big\|\, |f_i| - |g_i| \,\big\|_p < \infty \tag{2.8}$$

then$(f_i)_{i=1}^\infty$ is equivalent to $(g_i)_{i=1}^\infty$. This follows from (1.3). In particular, if $(f_i)_{i=1}^\infty$ is seminormalized unconditional basic in $L_p(\mathbb{R})$ then, by first approximating each $(f_i)_{i=1}^\infty$ by a simple dyadic function and then using the above consequence of (1.3), there exists a block basis $(g_i)_{i=1}^\infty$ of $(h_{(i,n)})_{i\in\mathbb{N},n\in\mathbb{Z}}$ satisfying (2.8) and thus being equivalent to $(f_i)_{i=1}^\infty$.

Proof of Theorem 2.11. By Proposition 2.13, it suffices to prove that X does not contain an isomorph of ℓ_2. By our above remarks we can choose a block basis $(g_i)_{i=1}^\infty$ of $(h_{(i,n)})$ which satisfies (2.8). In particular $(g_i)_{i=1}^\infty$ is equivalent to $(f_i)_{i=1}^\infty$ and we maintain

$$\sum_{i=1}^\infty \|g_i|_I\|_p^p < \infty \quad \text{for all bounded intervals} \quad I. \tag{2.9}$$

Thus we need only show that $[(g_i)_{i=1}^\infty]$ does not contain an isomorph of ℓ_2. If this is false, then there exists a normalized block basis $(\bar{g}_i)_{i=1}^\infty$ of $(g_i)_{i=1}^\infty$ which is equivalent to the unit vector basis of ℓ_2. Set $\bar{X} = [(\bar{g}_i)_{i=1}^\infty]$. By Lemma 2.12 there exists $M \in \mathbb{N}$ and $1 \le C < \infty$ so that for all $\bar{g} \in \bar{X}$, $I = [-M, M]$,

$$\|\bar{g}|_I\|_p \ge C^{-1}\|\bar{g}\|_p\ . \tag{2.10}$$

Since $(g_i)_{i=1}^\infty$ is a block basis of $(h_{(i,n)})$ then so is the normalized and unconditional sequence $(g_i|_I/\|g_i|_I\|_p)_{i=1}^\infty$. This yields lower ℓ_p and upper ℓ_2 estimates for this sequence and (g_i) (see (1.2)). From this and (2.10) we obtain for some constant $D < \infty$ and for all

$(a_i) \subseteq \mathbb{R}$, and $\bar{g} = \sum_{i=1}^{\infty} a_i g_i \in \bar{X}$,

$$\begin{aligned} D^{-1}\Big(\sum_{i=1}^{\infty} |a_i|^p\Big)^{1/p} &\le D^{-1/2}\Big\|\sum_{i=1}^{\infty} a_i g_i\Big\|_p \\ &\le \Big\|\sum_{i=1}^{\infty} a_i g_i|_I\Big\|_p = \Big\|\sum_{i=1}^{\infty} a_i\|g_i|_I\|_p \frac{g_i|_I}{\|g_i|_I\|_p}\Big\|_p \qquad (2.11)\\ &\le D\Big(\sum_{i=1}^{\infty} |a_i|^2\|g_i|_I\|_p^2\Big)^{1/2} . \end{aligned}$$

By (2.9) there exists $n_0 \in \mathbb{N}$ with

$$\Big(\sum_{i=n_0}^{\infty} \|g_i|_I\|_p^p\Big)^{1/p} < (2C)^{-1}D^{-2} . \tag{2.12}$$

Let $\bar{g}$ be an element of S_X which has the property that if we expand it in terms of the g_i's, i.e., if we write it as $\bar{g} = \sum_{i=1}^{\infty} a_i g_i$ then $a_j = 0$ for $j \le n_0$. From (2.10) and (2.11),

$$C^{-1} \le \|\bar{g}|_I\|_p \le D\Big(\sum_{i=n_0}^{\infty} a_i^2\|g_i|_I\|_p^2\Big)^{1/2} \le D\Big[\|(a_i^2)_{i=n_0}^{\infty}\|_{\ell_{p/2}} \cdot \|\big(\|g_i|_I\|_p^2\big)_{i=n_0}^{\infty}\|_{\ell_{\frac{p}{p-2}}}\Big]^{1/2}$$

(applying Hölder's inequality for $p/2$ and $p/p-2$)

$$= D\|(a_i)_{i=n_0}^{\infty}\|_{\ell_p}\Big(\sum_{i=n_0}^{\infty} \|g_i|_I\|_p^{\frac{2p}{p-2}}\Big)^{\frac{p-2}{2p}}$$

$$\le D^2\|\bar{g}|_I\|_p\Big(\sum_{i=n_0}^{\infty} \|g_i|_I\|_p^p\Big)^{1/p}$$

(by (2.11) and since $p \le 4$, $\frac{2p}{p-2} \ge p$)

$$\le (2C)^{-1} \quad \text{by (2.12)},$$

which is a contradiction. □

When $p > 4$ the possible structure is more complicated.

Theorem 2.14. *Let $4 < p < \infty$. There exists $f \in L_p(\mathbb{R})$ and $\Lambda \subseteq \mathbb{Z}$ so that $(f_{(\lambda)})_{\lambda\in\Lambda}$ is an unconditional basic sequence with $X_p(f,\Lambda)$ containing an isomorph of $L_p(\mathbb{R})$.*

Proof. We identify, in the canonical way, $L_p(\mathbb{R})$ with $(\bigoplus_{i\in\mathbb{Z}} L_p[0,1])_{\ell_p}$. Since $L_p[0,1]$ is isometrically isomorphic to $L_p([0,1]^2)$, we need only produce $f = (f_i)_{i\in\mathbb{Z}} \in (\bigoplus_{i\in\mathbb{Z}} L_p[0,1]^2)_{\ell_p}$ and $\Lambda \subseteq \mathbb{N}$ so that setting for $\lambda \in \Lambda$ $f_{(\lambda)} = (f_{i-\lambda})_{i\in\mathbb{Z}}$, then $X_p(f,\Lambda)$ contains an isomorph of $L_p[0,1]$ and $(f_{(\lambda)})_{\lambda\in\Lambda}$ is unconditional.

Letting, as before, $(h_n)_{n=1}^\infty$ be the normalized Haar basis for $L_p[0,1]$ and $(r_n)_{n=1}^\infty$ the Rademacher functions on $[0,1]$ we have, for some constants C_p and D_p (see (1.3)), for all $(a_i) \subseteq \mathbb{R}$,

$$\left(\sum_{i=1}^\infty |a_i|^2\right)^{1/2} \le \Big\|\sum_{i=1}^\infty a_i r_i\Big\|_p \le C_p\left(\sum_{i=1}^\infty |a_i|^2\right)^{1/2} \tag{2.13}$$

and

$$\Big\|\sum_{i=1}^\infty a_i h_i\Big\|_p \overset{D_p}{\sim} \Big\|\sum_{i=1}^\infty a_i^2 |h_i|^2\Big\|_{p/2}^{1/2} . \tag{2.14}$$

Since $p>4$ we can choose $(\varepsilon_i)_{i=1}^\infty \subseteq (0,1)$ so that

$$\sum_{i=1}^\infty \varepsilon_i^p = 1 \tag{2.15}$$

and there exists a partition $(J_n)_{n=1}^\infty$ of $\mathbb{N}$ into finite intervals with

$$\sum_{j\in J_n} \varepsilon_j^4 = 1 \quad \text{for all} \quad n\in\mathbb{N} . \tag{2.16}$$

We are ready to define $f=(f_i)_{i\in\mathbb{Z}} \in (\bigoplus_{i\in\mathbb{Z}} L_p[0,1]^2)_{\ell_p}$. Set for $i\in\mathbb{Z}$,

$$f_i = \begin{cases} \varepsilon_j h_n \otimes r_j \ , & \text{if} \ \ i=3^j \ \ \text{with} \ \ j\in J_n \ \ \text{for some} \ \ n\in\mathbb{N} \\ 0 \ , & \text{otherwise.} \end{cases} \tag{2.17}$$

where $h_n\otimes r_j$ is placed on the i^{th} copy of $[0,1]^2$. Note that

$$\|f\|_p^p = \sum_{n\in\mathbb{N}}\sum_{j\in J_n} \|\varepsilon_j h_n\otimes r_j\|_p^p = \sum_{n\in\mathbb{N}}\sum_{j\in J_n} \varepsilon_j^p = 1 \ .$$

Let $\Lambda=\{-3^j : j\in\mathbb{N}\}$ and so our translated sequence is $(f_{(-3^j)})_{j=1}^\infty$. For ease of notation below we shall write $f_{(-3^j)}$, f shifted 3^j units left, as $f^{(-3^j)}$, and $f^{(-3^j)} = (f_i^{(-3^j)})_{i\in\mathbb{Z}}$ where $f_i^{(-3^j)}$ denotes $f^{(-3^j)}$ restricted to the i^{th} $[0,1]^2$.

Now $f_0^{(-3^j)} = \varepsilon_j h_n \otimes r_j$ if $j \in J_n$ and so for $(a_j) \subseteq \mathbb{R}$,

$$\Big\|\sum_{j\in\mathbb{N}} a_j f_0^{(-3^j)}\Big\|_p^p = \Big\|\sum_{n\in\mathbb{N}}\sum_{j\in J_n} a_j\varepsilon_j h_n\otimes r_j\Big\|_p^p$$

$$= \int_0^1\int_0^1 \Big|\sum_{n\in\mathbb{N}}\sum_{j\in J_n} a_j\varepsilon_j h_n(s) r_j(t)\Big|^p\,dt\,ds$$

$$\overset{C_p^p}{\sim} \int_0^1 \Big|\sum_{n\in\mathbb{N}}\sum_{j\in J_n} a_j^2\varepsilon_j^2 h_n^2(s)\Big|^{p/2} ds\ ,\ \text{by (2.13)} \tag{2.18}$$

$$= \Big\|\sum_{n\in\mathbb{N}}\Big(\sum_{j\in J_n} a_j^2\varepsilon_j^2\Big)h_n^2\Big\|_{p/2}^{p/2}$$

$$\overset{D_p^p}{\sim} \Big\|\sum_{n\in\mathbb{N}}\Big(\sum_{j\in J_n} a_j^2\varepsilon_j^2\Big)^{1/2} h_n\Big\|_p^p\ ,\quad \text{by (2.14).}$$

Now for $j\in\mathbb{N}$, $f_\ell^{(-3^j)} \neq 0$ iff $\ell = 3^k - 3^j$ for some $k\in\mathbb{N}$. If $\ell\neq 0$ and $\ell = 3^k-3^j = 3^{k'}-3^{j'}$ for $k,k',j,j'\in\mathbb{N}$ then $k=k'$ and $j=j'$. Thus the functions $(f^{(-3^j)})_{j\in\mathbb{N}}$ are disjointly supported except on the 0^{th} copy of $[0,1]^2$. Also

$$\Big\|\sum_{\substack{\ell\neq 0\\ \ell\in\mathbb{Z}}} f_\ell^{(-3^j)}\Big\|_p^p = 1-\varepsilon_j^p\ ,$$

From this and (2.18) we obtain for some K, for all $(a_i)\subseteq\mathbb{R}$,

$$\Big\|\sum_{j\in\mathbb{N}} a_j f^{(-3^j)}\Big\|_p^p \overset{K}{\sim} \Big\|\sum_{n\in\mathbb{N}}\Big(\sum_{j\in J_n} a_j^2\varepsilon_j^2\Big)^{1/2} h_n\Big\|_p^p + \sum_{j\in\mathbb{N}} |a_j|^p\ . \tag{2.19}$$

Thus $(f^{(-3^j)})_{j=1}^\infty$ is unconditional and we shall next construct a block basis $(b^{(n)})_{n=1}^\infty$ of $(f^{(-3^j)})_{j=1}^\infty$ which is equivalent to $(h_n)_{n=1}^\infty$.

For $n\in\mathbb{N}$ set

$$b^{(n)} = \sum_{j\in J_n}\varepsilon_j f^{(-3^j)}\ .$$

From (2.19), for $(c_n)\subseteq\mathbb{R}$

$$\Big\|\sum_{n=1}^\infty c_n b^{(n)}\Big\|_p^p = \Big\|\sum_{n=1}^\infty c_n\sum_{j\in J_n}\varepsilon_j f^{(-3^j)}\Big\|_p^p$$

$$\overset{K^p}{\sim} \Big\|\sum_{n=1}^\infty c_n\Big(\sum_{j\in J_n}\varepsilon_j^4\Big)^{1/2} h_n\Big\|_p^p + \sum_{n=1}^\infty\sum_{j\in J_n}|c_n|^p\varepsilon_j^p$$

$$= \Big\|\sum_{n=1}^\infty c_n h_n\Big\|_p^p + \sum_{n=1}^\infty |c_n|^p\Big(\sum_{j\in J_n}\varepsilon_j^p\Big)\ .$$

Thus, using this and (1.2), the lower ℓ_p estimate of $(h_n)_{n=1}^\infty$, we see that $(b^{(n)})_{n=1}^\infty$ is equivalent to $(h_n)_{n=1}^\infty$. □

We next note that under certain additional assumptions we cannot have the situation of Theorem 2.14.

Proposition 2.15. *Let $4 < p < \infty$ and let $(f_i)_{i=1}^\infty$ be an unconditional basis for $X \subseteq L_p(\mathbb{R})$ where the f_i's are all translations of some fixed $f \in L_p(\mathbb{R})$. If either*

a) *$f \in L_2(\mathbb{R})$ or*
b) *$\sum_{n\in\mathbb{Z}} \|f|_{[n-1,n]}\|_p^{\frac{2p}{p-2}} < \infty$*

then X embeds isomorphically into ℓ_p.

Proof. b) follows easily from the proof of Theorem 2.11. Indeed we can use b) to deduce the next to last inequality in that proof, rather than using $p \le 4$ as was done there.

a) We assume the contrary so by Proposition 2.13 X contains an isomorph of ℓ_2. We choose I, $(g_i)_{i=1}^\infty$ and $(\bar{g}_i)_{i=1}^\infty$ as in the proof of Theorem 2.11 with the additional assumption that $\sum_{i=1}^\infty \big\| |f_i| - |g_i| \big\|_2 < \infty$.

Hence, using $f \in L_2(\mathbb{R})$,

$$\sum_{i=1}^{\infty} \|g_i|_I\|_2^2 < \infty \ . \tag{2.20}$$

Now $(\bar{g}_i|_I)_{i=1}^\infty$ is a block basis of $(h_{(i,n)})$ which is equivalent to the unit vector basis of ℓ_2. This forces $\|\cdot\|_p$ and $\|\cdot\|_2$ to be equivalent on $[(\bar{g}_i|_I)_{i=1}^\infty] \subseteq L_p(I)$ [KP]. $(\bar{g}_i)_{i=1}^\infty$ is also a normalized block basis of $(g_i)_{i=1}^\infty$ and so we may write $\bar{g}_i = \sum_{j=n_{i-1}+1}^{n_i} c_j g_j$ for some scalars (c_j), $n_0 < n_1 < \cdots$ and all $i \in \mathbb{N}$. Since $(g_i|_I)_{i=1}^\infty$ is also a block basis of $(h_{(i,n)})$ and hence is orthogonal in $L_2(I)$, we have for $i \in \mathbb{N}$,

$$\|\bar{g}_i|_I\|_2 = \Big\| \sum_{j=n_{i-1}+1}^{n_i} c_j g_i|_I \Big\|_2 = \Big(\sum_{j=n_{i-1}+1}^{n_i} c_j^2 \|g_j|_I\|_2^2 \Big)^{1/2} \le \sup_j |c_j| \Big(\sum_{j=n_{i-1}+1}^{n_i} \|g_i|_I\|_2^2 \Big)^{1/2}$$

and the latter converges to 0 as $i \to \infty$ by (2.20). Thus $\|\bar{g}_i|_I\|_2 \to 0$ so $\|\bar{g}_i|_I\|_p \to 0$ which is a contradiction. □

We next present two more examples. The first is an easy example of a translation sequence in L_p $(2 < p)$ which is unconditional but not equivalent to the ℓ_p basis and so Theorem 2.11 cannot be improved to get (f_i) equivalent to the unit vector basis of ℓ_p. The second is a translation sequence (f_i) in L_p, $p > 4$, which is basic but not unconditional.

Example 2.16. Let $2 < p < \infty$. There exists $f \in L_p(\mathbb{R})$ so that $(f_{(n)})_{n=1}^{\infty}$, the sequence of translations of f by $n \in \mathbb{N}$, is unconditional basic but not equivalent to the unit vector basis of ℓ_p.

Of course we already know this for $p > 4$ by Theorem 2.14. Let $(r_n)_{n\in\mathbb{Z}}$ be an enumeration of the Rademacher functions on $[0,1]$ extended trivially to functions defined on all of $\mathbb{R}$. We define $\tilde{r}_n(\cdot) = r_n((\cdot) - n)$, for $n \in \mathbb{Z}$, and let $f = \sum_{n\in\mathbb{Z}} \frac{\tilde{r}_n}{\sqrt{|n|}}$, where we regard $\frac{1}{\sqrt{|0|}} = 1$. Note that $\|f\|_p^p = 1 + 2\sum_{n=1}^{\infty} n^{-p/2} < \infty$, since $p > 2$. For $(a_i) \in c_{00}$, $g = \sum a_i f_{(i)}$ and $x \in [k, k+1]$, $k \in \mathbb{Z}$, we observe

$$g(x) = \sum_{i=1}^{\infty} a_i f(x - i) = \sum_{i=1}^{\infty} a_i \frac{r_{k-i}(x-k)}{\sqrt{|k-i|}} \ .$$

Thus, for some $c_p > 0$

$$\|g\|_p^p = \sum_{k\in\mathbb{Z}} \|g|_{[k,k+1]}\|_p^p \overset{c_p}{\sim} \sum_{k\in\mathbb{Z}} \left(\sum_{i=1}^{\infty} \frac{a_i^2}{|k-i|} \right)^{p/2} ,$$

which shows that $(f_{(i)})_{i=1}^{\infty}$ is unconditional. Moreover, if we let $a_i = 1$, for $i = 1, \ldots, m \in \mathbb{N}$ for $m \in \mathbb{N}$ we obtain

$$\Big\| \sum_{i=1}^{m} f_{(i)} \Big\|_p^p \geq c_p \sum_{k=1}^{m} \left(\sum_{i=1}^{m} \frac{1}{|m-i|} \right)^{p/2} \geq c_p \, m (\log m)^{p/2} \ .$$

Thus $(f_{(i)})$ is not equivalent to the unit vector basis of ℓ_p. □

Example 2.17. Let $p > 4$. There exists $f \in L_p(\mathbb{R})$ and $\Lambda \subset \mathbb{Z}$ so that $\{f_{(\lambda)} : \lambda \in \Lambda\}$ is basic in some order, but not unconditional.

As in Theorem 2.14 we identify $L_p(\mathbb{R})$ with $(\oplus_{n\in\mathbb{Z}} L_p[0,1])_p$, and we write f as $(f_i :\in \mathbb{Z})$ with $f_i \in L_p(0,1)$, for $i \in \mathbb{Z}$, and, as in Theorem 2.14, we write $f^{(\lambda)}$ instead of $f_{(\lambda)}$.

For $j \in \mathbb{N}$ let $a_j = j^{-1/4}$, and $a_0 = 1$. Let (r_j) be the Rademacher sequence on $[0,1]$. We define $f = (f_i)_{i\in\mathbb{Z}}$ by

$$f_i = \begin{cases} a_{j-1} r_j - a_{j+1} r_{j+1} & \text{if } i = 3^j \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Since $p > 4$, $(a_i) \in \ell_p$ and, thus, $f \in (\oplus_{n\in\mathbb{Z}} L_p[0,1])_p$. We let $\Lambda = \{-3^j : j \in \mathbb{N}\}$. For $(b_j)_{j=1}^n \subset \mathbb{R}$ we compute ($b_0 = 0$)

$$\begin{aligned}\sum_{j=1}^n b_j f_0^{(-3^j)} &= \sum_{j=1}^n b_j f_{3^j} \\ &= \sum_{j=1}^n b_j (a_{j-1} r_j - a_{j+1} r_{j+1}) \\ &= \sum_{j=1}^n (b_j a_{j-1} - b_{j-1} a_j) r_j - b_n a_{n+1} r_{n+1} \ .\end{aligned}$$

We deduce that

$$\begin{aligned}\Big\| \sum_{j=1}^n a_j f_0^{(-3^j)} \Big\|_p &= \|r_1 - a_n a_{n+1} r_{n+1}\|_p \to 1 \ \text{ if } \ n \to \infty \ , \quad \text{and} \\ \Big\| \sum_{j=1}^n (-1)^{j+1} a_j f_0^{(-3^j)} \Big\|_p &= \Big\| r_1 + \sum_{i=2}^n (-1)^{i+1} 2 a_{i-1} a_i r_i \pm a_n a_{n+1} r_{n+1} \Big\|_p \\ &\sim \Big(\sum_{i=1}^n a_i^4 \Big)^{1/2} = \Big(\sum_{i=1}^n \frac{1}{i} \Big)^{1/2} .\end{aligned}$$

We can now apply the same arguments as in the proof of Theorem 2.14 and obtain

$$\Big\| \sum b_j f^{(-3^j)} \Big\|_p \sim \Big\| \sum b_j f_0^{(-3^j)} \Big\|_p \vee \Big(\sum |b_j|^p \Big)^{1/p} .$$

From this expression it follows that $(f^{(-3^j)})_{j=1}^\infty$ is basic.

Indeed

$$\Big\| \sum_{j=1}^n b_j f^{(-3^j)} \Big\|_p \sim \Big(\sum_{j=1}^n (b_j a_{j-1} - b_{j-1} a_j)^2 + (b_n a_{n+1})^2 \Big)^{1/2} \vee \Big(\sum_{j=1}^n |b_j|^p \Big)^{1/p} .$$

Let the right hand expression be equal to 1 with

$$\sum_{j=1}^n (b_j a_{j-1} - b_{j-1} a_j)^2 + (b_n a_{n+1})^2 = 1.$$

Then if $(b_n a_{n+1})^2 \le 1/2$, for any extension $(b_i)_{i=1}^m$, $m > n$, the right hand expression is at least $1/\sqrt{2}$. If $(b_n a_{n+1})^2 \ge 1/2$ then $b_n > 1/2^{1/4}$ and so $(\sum_{j=1}^m |b_j|^p)^{1/p} \ge 2^{-1/4}$.

Finally $(f^{(-3^j)})_{j=1}^\infty$ is not unconditional since

$$\Big\| \sum_{j=1}^n a_j f^{(-3^j)} \Big\|_p \sim (\log n)^{1/p}$$

while

$$\Big\| \sum_{j=1}^n (-1)^{j+1} a_j f^{(-3^j)} \Big\|_p \sim (\log n)^{1/2} \ .$$

□

The translation problem can, of course, be considered in other rearrangement invariant function spaces on $\mathbb{R}$. We end this section with a simple result in the space $L_p(\mathbb{R}) \cap L_2(\mathbb{R})$ for $2 < p < \infty$. The norm is given by $\|f\| = \|f\|_p \vee \|f\|_2$ and the space is isomorphic to $L_p(\mathbb{R})$ (see e.g., [JMST] for more on this space).

Proposition 2.18. *Let $2 < p < \infty$ and let $(f_i)_{i=1}^\infty$ be an unconditional basis for $X \subseteq L_p(\mathbb{R}) \cap L_2(\mathbb{R})$ consisting of translations of some fixed $f \in L_p(\mathbb{R}) \cap L_2(\mathbb{R})$. Then $(f_i)_{i=1}^\infty$ is equivalent to the unit vector basis of ℓ_2.*

Proof. As before, by first carefully approximating in both $\|\cdot\|_p$ and $\|\cdot\|_2$ each f_i by a simple dyadic function $\tilde{f}_i$ and then choosing a block basis $(g_i)_{i=1}^\infty$ of $(h_{(i,n)})$ with $|g_i| = |\tilde{f}_i|$ for all i, we obtain: $(g_i)_{i=1}^\infty$ is equivalent to $(f_i)_{i=1}^\infty$ in $L_p(\mathbb{R}) \cap L_2(\mathbb{R})$.

Now $(g_i)_{i=1}^\infty$ is unconditional and semi-normalized in $L_p(\mathbb{R}) \cap L_2(\mathbb{R})$ which is isomorphic to L_p. Hence by (1.2), (g_i) admits an upper ℓ_2-estimate. Furthermore $(g_i)_{i=1}^\infty$ is unconditional and semi-normalized in $L_2(\mathbb{R})$ and thus also admits a lower ℓ_2-estimate in $\|\cdot\|_2$ and so in $L_p(\mathbb{R}) \cap L_2(\mathbb{R})$. □

3. Discrete versions of the problem

It remains open if $L_p(\mathbb{R})$, $1 < p < \infty$, admits a basis of translations of some fixed $f \in L_p(\mathbb{R})$ (see section 4 for more open problems). The examples in section 3 were all integer translations and this leads to a natural

Question 3.1. Let $1 < p < \infty$. Is there a set $\Lambda = \{\lambda_n : n \in \mathbb{N}\} \subseteq \mathbb{Z}$ and an $f \in L_p(\mathbb{R})$ so that that $(f_{(\lambda_n)} : n \in \mathbb{N})$ is a basis for $L_p(\mathbb{R})$?

The answer is no for $1 < p \leq 2$ (and of course for $p = 1$ by Theorem 1.7) by Theorem 1.3. We also deduce this as a Corollary to Proposition 3.7 below. The answer is also no for $\Lambda = \mathbb{Z}$ by [AO].

Proposition 3.2. [AO]. *Let $1 < p < \infty$. There is no $\lambda > 0$ and $f \in L_p(\mathbb{R})$ so that $\{f_{(\lambda n)} : n \in \mathbb{Z}\}$ can be ordered to be a basis for $L_p(\mathbb{R})$.*

Proof. We will prove a more general result below (see Proposition 3.7 and Corollary 3.12). □

We can do a bit better in L_1 for certain spaces $X_1(f, (\lambda n)_{n\in\mathbb{Z}})$. By Theorem 1.1, $X_1(f, \mathbb{R}) = L_1(\mathbb{R})$ forces $\hat{f}(t) \neq 0$ for all t.

Lemma 3.3. *Let $f \in L_1(\mathbb{R})$ with $\widehat{f}(t) \neq 0$ for all t, and let $\Lambda = \{\lambda_n : n \in \mathbb{N}\}$ be uniformly discrete. Then $\{f_{(\lambda_n)}\}_{n\in\mathbb{N}}$ is a non-fundamental minimal system in $L_1(\mathbb{R})$.*

Proof. We use the fact that *for a uniformly discrete Λ, there exists $a > 0$ so that $(e^{i\lambda_n t})_{n\in\mathbb{N}}$ is not complete in $C[-a,a]$*. As pointed out to us by J. Bruna, this follows from Paley-Wiener theory by constructing, from an entire function of finite exponential type, a Paley-Wiener function which vanishes on Λ. Alternately, this can be also quickly deduced from Beurling-Malliavin radius of completeness formula (cf. [Ko, section IX D]) and the fact that the uniformly discrete sequences have finite Beurling-Malliavin density. For convenience of the reader, we present a proof. We recall the definition of Beurling-Malliavin density D_{BM}. For $\Lambda \subset (0,\infty)$ and $D > 0$, a family of disjoint intervals (a_k, b_k), $0 < a_1 < b_1 < \ldots < a_k < b_k < \ldots \nearrow \infty$ is called substantial for D if

$$\frac{n_\Lambda(a_k,b_k)}{b_k - a_k} > D,\ k = 1,2,\ldots, \quad \sum_k \Big(\frac{b_k - a_k}{b_k}\Big)^2 = \infty,$$

where $n_\Lambda(a_k,b_k)$ is the number points of Λ in the interval (a_k,b_k). Then the density is defined by

$$D_{BM}(\Lambda) = \sup\{D > 0 : \text{there exists a substantial family for D}\}.$$

For a general Λ, put $D_{BM}(\Lambda) = \max\{D_{BM}(\Lambda^+), D_{BM}(\Lambda^-)\}$ where $\Lambda^+ = \Lambda \cap \mathbb{R}^+$, $\Lambda^- = (-\Lambda)\cap\mathbb{R}^+$. Beurling-Malliavin radius of completeness theorem asserts that $\{e^{i\lambda_n t} : \lambda_n \in \Lambda\}$ is complete in $C[-a,a]$ if and only if $\pi D_{BM}(\Lambda) \geq a$.

Now suppose that Λ is uniformly discrete and let $\delta = \inf\{|\lambda - \lambda'| : \lambda, \lambda' \in \Lambda, \lambda \neq \lambda'\} > 0$. Since $n_\Lambda(a_k,b_k)/(b_k - a_k) < 2/\delta$ for all $b_k > a_k > 0$, no $D > 2/\delta$ can be substantial for Λ, and therefore $D_{BM}(\Lambda) < 2/\delta$. Thus, by Beurling-Malliavin theorem, $(e^{i\lambda_n t})_{n\in\mathbb{N}}$ is not complete in $C[-b,b]$ for $b > 2/\delta$.

To see the minimality of $\{f_{(\lambda_n)}\}$, suppose to the contrary that for some n_0, $f_{\lambda_{n_0}} \in [(f_{\lambda_n})_{n\neq n_0}]$ in $L_1(\mathbb{R})$. Then $\hat{f}_{\lambda_{n_0}}(t) = \hat{f}(t)e^{-i\lambda_{n_0}t} \in [(\hat{f}(t)e^{-i\lambda_n t})_{n\neq n_0}]$ in $C_0(\mathbb{R})$. Now $\hat{f}(t) \neq 0$ for all t, so $e^{-i\lambda_{n_0}t} \in [\{e^{-i\lambda_n t} : n \neq n_0\}] \subset C[-b,b]$ for all $b > 0$. Thus $(e^{-i\lambda_n t})_{n\neq n_0}$ is complete in $C[-b,b]$ (cf. [Yo, Theorem 8, p. 129]). This contradicts the fact when $b > a$. Similarly, observe that $\{f_{(\lambda_n)}\}$ cannot be fundamental in $L_1(\mathbb{R})$, indeed otherwise $(e^{-i\lambda_n t})_{n\in\mathbb{N}}$ would be complete in $C[-b,b]$ for all $b > 0$. □

Note that the assumption $\hat{f}(t) \neq 0$ for all t is not frivolous due to Remark 2.8b).

Proposition 3.4. *Let $f \in L_1(\mathbb{R})$ with $\hat{f}(t) \neq 0$ for all t and let $\lambda > 0$. Then $X_1(f,(\lambda n)_{n\in\mathbb{Z}})$ embeds almost isometrically into ℓ_1.*

Proof. By Corollary 2.4 it suffices to show that $(f_{(\lambda n)})_{n\in\mathbb{Z}}$ is a bounded minimal system. By Lemma 3.3 it is a minimal system. Let $g(f) = 1$, $g(f_{(\lambda n)}) = 0$ for $n \in \mathbb{Z}\setminus\{0\}$, $g \in L_\infty(\mathbb{R})$. Then $(f_{(\lambda n)}, g_{(\lambda n)})_{n\in\mathbb{Z}}$ is a bounded minimal system. □

Proposition 3.2 generalizes to ℓ_p-sums of a separable infinite dimensional Banach space X. Define $\ell_p(X) = \ell_p(\mathbb{Z}, X) = (\bigoplus_{n\in\mathbb{Z}} X)_{\ell_p}$. For $F = (f_n : n \in \mathbb{Z}) \in \ell_p(X)$ and $k \in \mathbb{Z}$, let $F^{(k)}$ be F shifted right k times. Precisely, $F^{(k)} = (f_{n-k})_{n\in\mathbb{Z}}$.

Proposition 3.5. *Let X be a separable infinite dimensional Banach space, $1 \le p < \infty$. There does not exist $F \in \ell_p(\mathbb{Z}, X)$ so that $\{F^{(k)} : k \in \mathbb{Z}\}$ is a basis for $\ell_p(\mathbb{Z}, X)$ in some order.*

Proof. Let $1/p + 1/q = 1$ and assume for some F that $(F^{(n_i)})_{i=1}^\infty$ is a basis for $\ell_p(\mathbb{Z}, X)$ where $(n_i)_{i=1}^\infty$ is a reordering of $\mathbb{Z}$. Let $(G_i)_{i\in\mathbb{N}} \subseteq \ell_q(\mathbb{Z}, X^*)$ be the biorthogonal functionals to $(F^{(n_i)})_{i=1}^\infty$. Choose i_0 with $n_{i_0} = 0$ and set $G_{i_0} = G = (g_n)_{n\in\mathbb{Z}}$, with $g_n \in X^*$ for $n \in \mathbb{N}$.

For $n, m \in \mathbb{Z}$,

$$\langle F^{(n)}, G^{(m)}\rangle = \sum_{k\in\mathbb{Z}} \langle f_{k-n}, g_{k-m}\rangle = \sum_{k\in\mathbb{Z}} \langle f_{k+m-n}, g_k\rangle = \langle F^{(n-m)}, G_{i_0}\rangle = \delta_{(m,n)} \ .$$

Again, from the uniqueness of the biorthogonal functionals to a basis (for $\ell_p(\mathbb{Z}, X)$), we see that $G_i = G^{(n_i)}$ for all $i \in \mathbb{N}$.

Choose $j \in \mathbb{N}$ with

$$\left(\sum_{i=j+1}^\infty \|f_{-n_i}\|^p\right)^{1/p} \le \frac{1}{2\|G\|} \ .$$

Since X is infinite dimensional, there exists $x \in S_X$ with $g_{-n_i}(x) = 0$ for all $i \le j$. Set $H = (\delta_{(0,n)} x : n \in \mathbb{Z}) \in \ell_p(\mathbb{Z}, X)$. Then

$$H = \sum_{i=1}^\infty \langle H, G^{(n_i)}\rangle F^{(n_i)} = \sum_{i=j+1}^\infty \langle H, G^{(n_i)}\rangle F^{(n_i)} \ .$$

Hence,

$$1 = \|x\| = \|H\| = \Big\| \sum_{i=j+1} \langle H, G^{(n_i)}\rangle F^{(n_i)} \Big\| = \sum_{i=j+1}^\infty \langle g_{-n_i}, f_{-n_i}\rangle$$

$$\le \sum_{i=j+1}^\infty \|g_{-n_i}\| \, \|f_{-n_i}\| \le \|G\| \left(\sum_{i=j+1}^\infty \|f_{-n_i}\|^p\right)^{1/p} \le \frac{1}{2} \ ,$$

a contradiction. □

Problem 3.6. Let $2 < p < \infty$ and let X be a Banach space with $\dim X \ge 2$. Does there exist $F \in \ell_p(\mathbb{Z}, X)$ and $(\lambda_i : i \in \mathbb{N}) \subseteq \mathbb{Z}$ so that $(F^{(\lambda_i)})_{i=1}^\infty$ is a basis for $\ell_p(\mathbb{Z}, X)$? What if $\dim X = 2$ or if $X = \ell_p$?

We do not ask the question for $p \leq 2$ because of the following proposition which generalizes Proposition 3.5 in that case.

Proposition 3.7. *Let $1 \leq p \leq 2$ and let X be a Banach space with $\dim(X) \geq 2$. Let $F = (f_i : i \in \mathbb{Z}) \in \ell_p(\mathbb{Z}, X)$. Then $[\{F^{(n)} : n \in \mathbb{Z}\}] \neq \ell_p(\mathbb{Z}, X)$.*

Corollary 3.8. [AO]. *Let $1 < p \leq 2$, $f \in L_p(\mathbb{R})$ and $\lambda > 0$. Then $[f_{(\lambda n)} : n \in \mathbb{Z}\}]$ is a proper subspace of $L_p(\mathbb{R})$. In particular, no subsequence of $\{f_{(\lambda n)} : n \in \mathbb{Z}\}$ can be ordered to form a basis for $L_p(\mathbb{R})$.*

Proof. We let F denote the Fourier transform on $L_1(\mathbb{R})+L_2(\mathbb{R})$ into the space of measurable functions on $\mathbb{R}$. F is a bounded linear operator, restricted to $L_1(\mathbb{R})$ (into $C_0(\mathbb{R})$) and when restricted to $L_2(\mathbb{R})$ (into $L_2(\mathbb{R})$). By the Riesz-Thorin interpolation theorem, F is also bounded as a linear operator from $L_p(\mathbb{R})$ into $L_q(\mathbb{R})$ $(1/p + 1/q = 1)$. Now since $L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \subseteq L_p(\mathbb{R})$, $F(L_p(\mathbb{R}))$ is dense in $L_q(\mathbb{R})$. For $f \in L_p(\mathbb{R})$ and $s \in \mathbb{R}$ we have $F(f_s) = e^{-is(\cdot)}F(f)$. Indeed for $f \in L_1(\mathbb{R})$ and $t \in \mathbb{R}$,

$$\begin{aligned} F(f_s)(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixt} f(x-s)\, dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(u+s)t} f(u)\, dx = e^{-is} F(f)(t) \ . \end{aligned}$$

For a general $f \in L_p(\mathbb{R})$ the result follows by the standard density argument.

Let $f \in L_p(\mathbb{R})$ and $\lambda \in \mathbb{R}$. If $[\{f_{(\lambda n)} : n \in \mathbb{Z}\}] = L_p(\mathbb{R})$ then $[\{e^{in\lambda(\cdot)}F(f) : n \in \mathbb{Z}\}] = L_q(\mathbb{R})$. This implies that $F(f) \neq 0$ a.e. and that $[\{e^{in\lambda(\cdot)} : n \in \mathbb{Z}\}] = L_q(|F(f)|^q\, dx)$. This in turn implies that all elements g of $L_p(|F(f)|^q\, dx)$ are λ-periodic ($g(x) - g(x+\lambda) = 0$ a.e.), a contradiction. $\square$

Remark 3.9. For $2 < p < \infty$ it is shown in [AO] (Theorem 1.2 above) that there exists $f \in L_p(\mathbb{R})$ so that $[(f^{(n)})_{n\in\mathbb{Z}}] = L_p(\mathbb{R})$.

We will use the Fourier transform on the abelian group $\mathbb{Z}$ (see [Ru]) and also assume our spaces to be defined over the complex field. For $x = (\xi_j) \in \ell_1(\mathbb{Z})$ we let $\widehat{x}$ be the function

$$\widehat{x} : [-\pi, \pi] \to \mathbb{R} \ , \quad \widehat{x}(t) = \sum_{n\in\mathbb{Z}} \xi_n e^{int} \ .$$

It is easy to see that $\widehat{x} \in C(T)$ when $x \in \ell_1(\mathbb{Z})$ (identifying, as usual, the torus T with $[-\pi, \pi]$ by identifying π and $-\pi$). Also the map

$$\widehat{(\cdot)} : \ell_1(\mathbb{Z}) \to C(T) \ , \qquad x \to \hat{x}$$

is a bounded linear operator of norm 1. For any $x = (\xi_n : n \in \mathbb{Z})$

$$\|\hat{x}\|_2^2 = \int_{-\pi}^{\pi} \Big| \sum_{n\in\mathbb{Z}} \xi_n e^{int} \Big|^2 \, dt = \int_{-\pi}^{\pi} \sum_{m,n\in\mathbb{Z}} \bar{\xi}_m \xi_n e^{i(n-m)t} \, dt = 2\pi \sum_{n\in\mathbb{Z}} |\xi_n|^2 \ .$$

Thus $\widehat{(\cdot)}$ extends to an isometry from $\ell_2(\mathbb{Z})$ to $L_2(T, \frac{1}{2\pi}\, dx)$.

Again, by Riesz-Thorin interpolation, the Fourier transform is a bounded linear operator from $\ell_p(\mathbb{Z})$ into $L_q(T)$ for $1 \le p \le 2$, $1/p + 1/q = 1$.

Since $\{\widehat{x} : x \in \ell_1(\mathbb{Z})\}$ is dense in $L_2(T)$, it follows that the image under the Fourier transform of $\ell_p(\mathbb{Z})$ is dense in $L_q(T)$.

We also need two lemmas before proving Proposition 3.7. The first is an easy exercise in real analysis.

Lemma 3.10. *Let $\nu \ll \mu$ be two σ-finite measures on the measure space (Ω, Σ). Then for $1 \le p < \infty$, if $D \subseteq L_p(\nu) \cap L_p(\mu)$ is dense in $L_p(\mu)$, it is also dense in $L_p(\nu)$.*

Proof. Let ρ be the Radon-Nikodym density of ν with respect to μ. For $n \in \mathbb{N}$ set

$$A_n = \left\{ \omega \in \Omega : \frac{1}{n} \le \rho(\omega) \le n \right\} \ .$$

For $n \in \mathbb{N}$, it follows that $L_p(\mu|_{A_n}) = L_p(\nu|_{A_n})$. Also by canonically identifying $L_p(\nu|_{A_n})$ with a subspace of $L_p(\nu)$, $\bigcup_{n\in\mathbb{N}} L_p(\nu|_{A_n})$ is dense in $L_p(\nu)$ and this yields the results. $\square$

Lemma 3.11. *Let $1 \le p \le 2$ and let $x = (\xi_n : n \in \mathbb{Z}) \in \ell_p(\mathbb{Z})$. Then $[(x^{(2n)})_{n\in\mathbb{Z}}] \neq \ell_p(\mathbb{Z})$.*

Proof. Recall $x^{(n)} = (\xi_{j-n} : j \in \mathbb{Z})$, for $n \in \mathbb{Z}$. For $n \in \mathbb{N}$, $t \in T$ and $z = (\zeta_j : j \in \mathbb{Z}) \in \ell_1(\mathbb{Z})$ we have

$$\widehat{z^{(n)}}(t) = \sum_{j\in\mathbb{Z}} \zeta_{j-n} e^{ijt} = \sum_{\ell\in\mathbb{Z}} \zeta_\ell e^{i(\ell+n)t} = e^{int} \widehat{z} \ .$$

By a density argument, we see that for any $x \in \ell_p(\mathbb{Z})$ and $n \in \mathbb{Z}$, $\widehat{x^{(n)}} = e^{in(\cdot)} \widehat{x}$.

Assume, to the contrary, that $[(x^{(2n)})_{n\in\mathbb{Z}}] = \ell_p(\mathbb{Z})$. It then follows that $[\{e^{i2n(\cdot)}\widehat{x} : n \in \mathbb{Z}\}] = L_q(T)$ and thus $\widehat{x} \neq 0$ a.e. Also that

$$[\{e^{i2n(\cdot)} : n \in \mathbb{Z}\}] = L_q(T, |\widehat{x}|^q \, dt) \ .$$

By Lemma 3.10, this implies that

$$[\{e^{i2n(\cdot)} : n \in \mathbb{Z}\}] = L_q(T) \ .$$

Since for any $n \in \mathbb{Z}$,

$$2\pi\langle e^{i2n(\cdot)}, \chi_{[-\pi,0]} - \chi_{[0,\pi]}\rangle = \int_{-\pi}^{0} e^{i2nt}dt - \int_{0}^{\pi} e^{i2nt}dt$$

$$= \int_0^\pi (e^{-i2nt} - e^{i2nt})\,dt = \begin{cases} 0\ , & \text{if } n = 0 \\ -2\displaystyle\int_0^\pi \sin(2nt)dt = 0\ , & \text{if } n \neq 0, \end{cases}$$

this cannot be true. □

Proof of Proposition 3.7. After projecting X onto ℓ_p^2 we see that we may assume $X = \ell_p^2$. Let I be the obvious isometry between $\ell_p(\mathbb{Z}, X)$ and $\ell_p(\mathbb{Z})$ denoted

$$(x_j)_{j\in\mathbb{Z}} \longmapsto (y_j)_{j\in\mathbb{Z}}$$

where if $x_j = (x_{(j,1)}, x_{(j,2)}) \in \ell_p^2$ then

$$y_{2j} = x_{(j,1)}, \qquad y_{2j+1} = x_{(j,2)}\ .$$

Then for $(x_j)_{j\in\mathbb{Z}} \in \ell_p(\ell_p^2)$, $(x^{(n)})_{n\in\mathbb{Z}} = (I(x)^{2n})_{n\in\mathbb{Z}}$ and the result follows from Lemma 3.11.

□

Remark. As noted above by the results of [AO] in section 4 we cannot hope to prove that given $f \in L_p(\mathbb{R})$, $2 < p < \infty$, $[(f^{(n)})_{n\in\mathbb{Z}}] \neq L_p(\mathbb{R})$. Nevertheless, by dualizing Proposition 3.7, we have the following

Corollary 3.12. *Let X be a Banach space with* $\dim(X) \geq 2$ *and let* $2 \leq p < \infty$. *Let* $F = (f_i)_{i\in\mathbb{Z}} \in \ell_p(\mathbb{Z}, X)$. *Then* $\{F^{(n)} : n \in \mathbb{Z}\}$ *is not a basis for* $\ell_p(\mathbb{Z}, X)$ *under any ordering.*

Proof. Assume that $F \in (f_i)_{i\in\mathbb{Z}} \in \ell_p(\mathbb{Z}, X)$ and that $(n_s)_{s\in\mathbb{N}}$ is an ordering of $\mathbb{Z}$ so that $(F^{(n_s)})_{s=1}^\infty$ is a basis for $\ell_p(\mathbb{Z}, X)$. Let $(G_s)_{s=1}^\infty \subseteq \ell_q(\mathbb{Z}, X^*)$ be the biorthogonal functionals of $(F^{(n_s)})_{s=1}^\infty$. Set $G = (g_j)_{j\in\mathbb{Z}} = G_1$. We let $G^{(m)} = (g_{j-m})_{j\in\mathbb{Z}}$, as usual. For $s, t \in \mathbb{N}$ and $m \in \mathbb{Z}$ we have

$$\langle F^{(n_s)}, G^{(n_t)}\rangle = \sum_{j\in\mathbb{Z}} \langle f_{j-n_s}, g_{j-n_t}\rangle = \sum_{k\in\mathbb{Z}} \langle f_{k+n_t-n_s}, g_k\rangle$$

$$= \langle F^{(n_s-n_t)}, G_1\rangle = \begin{cases} 1\ , & \text{if } n_s - n_t = n_1 \\ 0\ , & \text{if } n_s - n_t \neq n_1. \end{cases}$$

As before, we see that $G_s = G^{(n_s-n_1)}$. In particular, $\operatorname{span}\{G^{(n)} : n \in \mathbb{Z}\}$ is w^*-dense in $\ell_q(X^*)$. Let E be a two dimensional subspace of X and let P be a projection of X onto E. Let $Q : \ell_p(\mathbb{Z}, X) \to \ell_p(\mathbb{Z}, E)$ be the projection given by $Q(H) = (P(h_i))_{i\in\mathbb{Z}}$. It follows that

$\mathrm{span}(G^{(n)}|_{\ell_p(\mathbb{Z},E)})_{n\in\mathbb{Z}}$ is w^* dense in $\ell_q(\mathbb{Z},E^*)$ and hence norm dense (the latter is reflexive). This contradicts Proposition 3.7. $\square$

4. Results from the literature and open problems

We first cite some more known results from the literature.

Theorem 4.1. [DH, Theorem 5.1(b)]. *Let $g^{(1)}, g^{(2)}, \ldots, g^{(m)} \in L_2(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)$ and let $\Gamma_1, \Gamma_2, \ldots, \Gamma_m \subset \mathbb{R}^d$ be countable. Then $\{g^{(i)}_{(\lambda)} : i = 1, 2, \ldots, m,\ \lambda \in \Gamma_i\}$ cannot be ordered to be a Schauder basis of $L_2(\mathbb{R}^d)$.*

Theorem 4.2. ([ER] and [Ro], cf. [H, Theorem 9.18]) *If $g \in L_p(\mathbb{R}^d)$, $g \neq 0$, and $1 \leq p \leq \frac{2d}{d-1}$ then the functions $\{g((\cdot)-\alpha_k) : k = 1, 2, \ldots, N\}$ are linearly independent for any $N \in \mathbb{N}$ and any collection $(\alpha_k)_{k=1}^N \subseteq \mathbb{R}^d$ of distinct points.*

If $\frac{2d}{d-1} < p \leq \infty$, then for $N \in \mathbb{N}$ there exists $0 \neq g \in L_p(\mathbb{R}^d)$ and distinct points $(\alpha_k)_{k=1}^N \subseteq \mathbb{R}^d$ so that $\{g((\cdot)-\alpha_k) : k = 1, 2, \ldots, N\}$ is linearly dependent.

Our last cited result requires some notation. For $\Lambda \subseteq \mathbb{R}$ let $\mathcal{E}(\Lambda) = \mathrm{span}\{e^{i\lambda(\cdot)} : \lambda \in \Lambda\}$. Let $R(\Lambda) = \sup\{\rho > 0 : \mathcal{E}(\Lambda) \text{ is dense in } C[-\rho,\rho]\}$. Recall, $\Lambda \subseteq \mathbb{R}$ is *discrete* if it has no accumulation points.

Theorem 4.3. [BOU, Theorem 1]. *Let $\Lambda \subseteq \mathbb{R}$ be discrete. There exists $f \in L_1(\mathbb{R})$ so that $[\{f_{(\lambda)} : \lambda \in \Lambda\}] = L_1(\mathbb{R})$ if and only if $R(\Lambda) = \infty$.*

Finally we list some problems that remain open. The main one is

Problem 4.4. Let $1 < p < \infty$. Does there exist $f \in L_p(\mathbb{R})$ and $\Lambda \subseteq \mathbb{R}$ so that $\{f_{(\lambda)} : \lambda \in \Lambda\}$ can be ordered to be a basis for $L_p(\mathbb{R})$? Can we find f and a uniformly discrete set Λ so that $\{f_{(\lambda)} : \lambda \in \Lambda\}$ can be ordered to be a frame for $L_p(\mathbb{R})$?

By identifying $L_p(\mathbb{R})$ with $L_p[0,1]$ we have a more general version of the basis problem in 4.4.

Problem 4.5. Does there exist a normalized basis $(f_n)_{n=1}^\infty$ for $L_p[0,1]$, $1 < p < \infty$, so that for all $0 < b < 1$,

$$\sum_{n=1}^{\infty} \|f_n|_{[0,b]}\|^p < \infty \ ?$$

If $4 < p < \infty$ can we find such f_n's which form an unconditional basis for $L_p[0,1]$?

Problem 4.6. Let $4 < p < \infty$. Does there exist $f \in L_p(\mathbb{R})$ and $\Lambda \subseteq \mathbb{R}$ so that $(f_{(\lambda)})_{\lambda\in\Lambda}$ is an unconditional basis for $L_p(\mathbb{R})$?

We can also raise questions asking for less and here is one such question.

Problem 4.7. Let $1 < p < 4$. Does there exist $f \in L_p(\mathbb{R})$ and a uniformly discrete set $\Lambda \subseteq \mathbb{R}$ so that $[\{f_{(\lambda)} : \lambda \in \Lambda\}] \subseteq L_p(\mathbb{R})$ contains an isomorph of ℓ_2 and $(f_{(\lambda)})_{\lambda\in\Lambda}$ can be ordered to be a basic sequence (or a frame)?

Problem 4.8. Let $\Lambda \subseteq \mathbb{R}$ be uniformly discrete and $f \in L_1(\mathbb{R})$. Does $X_1(f, \Lambda)$ embed into ℓ_1?

References

[AOd] D. Alspach and E. Odell, *L_p spaces*, in "Handbook of Geometry of Banach Spaces", Vol.1, W.B. Johnson and J. Lindenstrauss, eds., Elsevier, Amsterdam (2001), 123–159.

[AO] A. Atzmon and A. Olevskii, *Completeness of integer translates in function spaces on $\mathbb{R}$*, J. of Approx. Theory **87** (1996) 291–327.

[BOU] J. Bruna, A. Olevskii and A. Ulanovskii, *Completeness in $L_1(\mathbb{R})$ of discrete translates*, Rev. Mat. Iberoamericana **22** no. 1, (2006), 1–16.

[CDOSZ] P.G. Casazza, S.J. Dilworth, E. Odell, Th. Schlumprecht and A. Zsák, *Coefficient quantization for frames in Banach spaces*, J. Math. Anal. Appl. **348** (2008), 66–86.

[CHL] P.G. Casazza, D. Han and D.R. Larson, *Frames for Banach spaces, the functional and harmonic analysis of wavelets and frames*, (San Antonio, TX, 1999), Contemp. Math. **247** (1999), 149–182.

[DH] B. Deng and C. Heil, *Density of Gabor Schauder bases*, in: Wavelet Applications in Signal and Image Processing VIII (San Diego, CA, 2000), A. Aldroubi, A. Lane, and M. Unser, eds., Proc. SPIE **4119**, SPIE, Bellingham, WA, (2000), 153–164.

[ER] G. Edgar and J. Rosenblatt, *Difference equations over locally compact abelian groups*, Trans. Amer. Math. Soc. **253** (1979), 273–289.

[H] C. Heil, *Linear independence of finite Gabor systems* in "Harmonic Analysis and Applications," A volume in honor of John J. Benedetto, Birkhauser, Boston, (2006), 171–206.

[J] W.B. Johnson, *On quotients of L_p which are quotients of ℓ_p*, Compositio Math. **34** (1) (1977), 69–89.

[JL] W.B. Johnson and J. Lindenstrauss, *Basic concepts in the geometry of Banach spaces*, in "Handbook of Geometry of Banach Spaces", Vol.1, W.B. Johnson and J. Lindenstrauss, eds., Elsevier, Amsterdam (2001), 1–84.

[JO] W.B. Johnson and E. Odell, *Subspaces of L_p which embed into ℓ_p*, Compos. Math. **28** (1974), 37 –49.

[JMST] W.B. Johnson, B. Maurey, G. Schechtman and L. Tzafriri, *Symmetric structures in Banach spaces*, Mem. Amer. Math. Soc. **19**(217) (1979) v+298.

[KP] M.I. Kadets and A. Pełczyński, *Bases, lacunary sequences and complemented subspaces in the spaces L_p*, Studia Math. **21** (1961/62), 161–176.

[KW] N.J. Kalton and D. Werner, *Property (M), M-ideals, and almost isometric structure of Banach spaces*, J. Reine Angew. Math. **461** (1995), 137–178.

[Ko] P. Koosis, *The logarithmic integral II*, Cambridge Studies in Advanced Mathematics, 21. Cambridge University Press, Cambridge, 1992.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I: Sequence Spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete 92, Springer-Verlag, Berlin (1977).

[Ol] A. Olevskii, *Completeness in $L_2(\mathbb{R})$ of almost integer translates*, C. R. Acad. Sci. Paris **324** (1997), 987–991.

[OZ] T. E. Olson and R. A. Zalik, *Nonexistence of a Riesz basis of translates, in: Approximation Theory*, Lecture Notes in Pure and Applied Math., **138**, Dekker, New York (1992) 401–408.

[R] J. Rosenblatt, private communication.

[Ro] J. Rosenblatt, *Linear independence of translations*, J. Austral. Math. Soc. (Series A) **59** (1995), 131-133.

[Ru] W. Rudin, *Fourier Analysis on Groups*, Interscience Publisher, John Wiley & sons (1967).

[S] G. Schechtman, *A remark on unconditional basic sequences in L_p $(1 < p < \infty)$*, Israel J. Math. **19** (1974), 220–224.

[Wi] N. Wiener, *The Fourier integral and certain of its applications*, Cambridge University Press, Cambridge, 1933 reprint: Dover, New York, 1958.

[Yo] R. M. Young, *An introduction to nonharmonic Fourier series*, Pure and Applied Mathematics, 93. Academic Press, Inc., New York-London, 1980.

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